a

2

b

Preface

This work blends together classic inequality results with brand new problems, some of which devised only a few days ago. What could be special about it when so many inequality problem books have already been written? We strongly believe that even if the topic we plunge into is so general and popular our book is very different. Of course, it is quite easy to say this, so we will give some supporting arguments. This book contains a large variety of problems involving inequalities, most of them difficult, questions that became famous in competitions because of their beauty and difficulty. And, even more importantly, throughout the text we employ our own solutions and propose a large number of new original problems. There are memorable problems in this book and memorable solutions as well. This is why this work will clearly appeal to students who are used to use Cauchy-Schwarz as a verb and want to further improve their algebraic skills and techniques. They will find here tough problems, new results, and even problems that could lead to research. The student who is not as keen in this field will also be exposed to a wide variety of moderate and easy problems, ideas, techniques, and all the ingredients leading to a good preparation for mathematical contests. Some of the problems we chose to present are known, but we have included them here with new solutions which show the diversity of ideas pertaining to inequalities. Anyone will find here a challenge to prove his or her skills. If we have not convinced you, then please take a look at the last problems and hopefully you will agree with us.

Finally, but not in the end, we would like to extend our deepest appreciation to the proposers of the problems featured in this book and to apologize for not giving all complete sources, even though we have given our best. Also, we would like to thank Marian Tetiva, Dung Tran Nam, Constantin Tănăsescu, Călin Popa and Valentin Vornicu for the beautiful problems they have given us and for the precious comments, to Cristian Babă, George Lascu and Călin Popa, for typesetting and for the many pertinent observations they have provided.

The authors

Contents

CHAPTER 1

Problems

1. Prove that the inequality

$$\sqrt{a^2+(1-b)^2}+\sqrt{b^2+(1-c)^2}+\sqrt{c^2+(1-a)^2}\geq\frac{3\sqrt{2}}{2}$$

holds for arbitrary real numbers a, b, c.

Kömal

2. [Dinu Şerbănescu] If $a, b, c \in (0, 1)$ prove that

$$\sqrt{abc}+\sqrt{(1-a)(1-b)(1-c)}<1.$$

Junior TST 2002, Romania

3. [Mircea Lascu] Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{b+c}{\sqrt{a}}+\frac{c+a}{\sqrt{b}}+\frac{a+b}{\sqrt{c}}\geq\sqrt{a}+\sqrt{b}+\sqrt{c}+3.$$

Gazeta Matematică

4. If the equation $x^4 + ax^3 + 2x^2 + bx + 1 = 0$ has at least one real root, then $a^2 + b^2 \geq 8$.

Tournament of the Towns, 1993

5. Find the maximum value of the expression $x^3+y^3+z^3-3xyz$ where $x^2+y^2+z^2 = 1$ and x, y, z are real numbers.

6. Let a, b, c, x, y, z be positive real numbers such that $x + y + z = 1$. Prove that

$$ax+by+cz+2\sqrt{(xy+yz+zx)(ab+bc+ca)}\leq a+b+c.$$

Ukraine, 2001

7. [Darij Grinberg] If a, b, c are three positive real numbers, then

$$\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2}\geq\frac{9}{4(a+b+c)}.$$

8. [Hojoo Lee] Let $a, b, c \geq 0$. Prove that

$$\sqrt{a^4+a^2b^2+b^4}+\sqrt{b^4+b^2c^2+c^4}+\sqrt{c^4+c^2a^2+a^4}\geq$$
$$\geq a\sqrt{2a^2+bc}+b\sqrt{2b^2+ca}+c\sqrt{2c^2+ab}.$$

Gazeta Matematică

9. If a, b, c are positive real numbers such that $abc = 2$, then

$$a^3+b^3+c^3\geq a\sqrt{b+c}+b\sqrt{c+a}+c\sqrt{a+b}.$$

When does equality hold?

JBMO 2002 Shortlist

10. [Ioan Tomescu] Let $x, y, z > 0$. Prove that

$$\frac{xyz}{(1+3x)(x+8y)(y+9z)(z+6)} \leq \frac{1}{7^4}.$$

When do we have equality?

Gazeta Matematică

11. [Mihai Piticari, Dan Popescu] Prove that

$$5(a^2+b^2+c^2) \leq 6(a^3+b^3+c^3)+1,$$

for all $a, b, c > 0$ with $a+b+c=1$.

12. [Mircea Lascu] Let $x_1, x_2, \ldots, x_n \in R$, $n \geq 2$ and $a > 0$ such that $x_1 + x_2 + \ldots + x_n = a$ and $x_1^2 + x_2^2 + \ldots + x_n^2 \leq \frac{a^2}{n-1}$. Prove that $x_i \in \left[0, \frac{2a}{n}\right]$, for all $i \in \{1, 2, \ldots, n\}$.

13. [Adrian Zahariuc] Prove that for any $a, b, c \in (1,2)$ the following inequality holds

$$\frac{b\sqrt{a}}{4b\sqrt{c}-c\sqrt{a}} + \frac{c\sqrt{b}}{4c\sqrt{a}-a\sqrt{b}} + \frac{a\sqrt{c}}{4a\sqrt{b}-b\sqrt{c}} \geq 1.$$

14. For positive real numbers a, b, c such that $abc \leq 1$, prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq a+b+c.$$

15. [Vasile Cîrtoaje, Mircea Lascu] Let a, b, c, x, y, z be positive real numbers such that $a+x \geq b+y \geq c+z$ and $a+b+c = x+y+z$. Prove that $ay+bx \geq ac+xz$.

16. [Vasile Cîrtoaje, Mircea Lascu] Let a, b, c be positive real numbers so that $abc = 1$. Prove that

$$1 + \frac{3}{a+b+c} \geq \frac{6}{ab+ac+bc}.$$

Junior TST 2003, Romania

17. Let a, b, c be positive real numbers. Prove that

$$\frac{a^3}{b^2} + \frac{b^3}{c^2} + \frac{c^3}{a^2} \geq \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}.$$

JBMO 2002 Shortlist

18. Prove that if $n > 3$ and $x_1, x_2, \ldots, x_n > 0$ have product 1, then

$$\frac{1}{1+x_1+x_1x_2} + \frac{1}{1+x_2+x_2x_3} + \cdots + \frac{1}{1+x_n+x_nx_1} > 1.$$

Russia, 2004

19. [Marian Tetiva] Let x, y, z be positive real numbers satisfying the condition

$$x^2 + y^2 + z^2 + 2xyz = 1.$$

Prove that

a) $xyz \leq \frac{1}{8}$;

b) $x + y + z \leq \frac{3}{2}$;

c) $xy + xz + yz \leq \frac{3}{4} \leq x^2 + y^2 + z^2$;

d) $xy + xz + yz \leq \frac{1}{2} + 2xyz$.

20. [Marius Olteanu] Let $x_1, x_2, x_3, x_4, x_5 \in R$ so that $x_1 + x_2 + x_3 + x_4 + x_5 = 0$. Prove that

$$|\cos x_1| + |\cos x_2| + |\cos x_3| + |\cos x_4| + |\cos x_5| \geq 1.$$

Gazeta Matematică

21. [Florina Cârlan, Marian Tetiva] Prove that if $x, y, z > 0$ satisfy the condition

$$x + y + z = xyz$$

then $xy + xz + yz \geq 3 + \sqrt{x^2+1} + \sqrt{y^2+1} + \sqrt{z^2+1}$.

22. [Laurenţiu Panaitopol] Prove that

$$\frac{1+x^2}{1+y+z^2} + \frac{1+y^2}{1+z+x^2} + \frac{1+z^2}{1+x+y^2} \geq 2,$$

for any real numbers $x, y, z > -1$.

JBMO, 2003

23. Let $a, b, c > 0$ with $a + b + c = 1$. Show that

$$\frac{a^2+b}{b+c} + \frac{b^2+c}{c+a} + \frac{c^2+a}{a+b} \geq 2.$$

24. Let $a, b, c \geq 0$ such that $a^4 + b^4 + c^4 \leq 2(a^2b^2 + b^2c^2 + c^2a^2)$. Prove that

$$a^2 + b^2 + c^2 \leq 2(ab + bc + ca).$$

Kvant, 1988

25. Let $n \geq 2$ and $x_1, ..., x_n$ be positive real numbers satisfying

$$\frac{1}{x_1 + 1998} + \frac{1}{x_2 + 1998} + ... + \frac{1}{x_n + 1998} = \frac{1}{1998}.$$

Prove that

$$\frac{\sqrt[n]{x_1 x_2 ... x_n}}{n-1} \geq 1998.$$

Vietnam, 1998

26. [Marian Tetiva] Consider positive real numbers x, y, z so that

$$x^2 + y^2 + z^2 = xyz.$$

Prove the following inequalities

a) $xyz \geq 27$;

b) $xy + xz + yz \geq 27$;

c) $x + y + z \geq 9$;

d) $xy + xz + yz \geq 2(x + y + z) + 9$.

27. Let x, y, z be positive real numbers with sum 3. Prove that

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \geq xy + yz + zx.$$

Russia, 2002

28. [D. Olteanu] Let a, b, c be positive real numbers. Prove that

$$\frac{a+b}{b+c} \cdot \frac{a}{2a+b+c} + \frac{b+c}{c+a} \cdot \frac{b}{2b+c+a} + \frac{c+a}{a+b} \cdot \frac{c}{2c+a+b} \geq \frac{3}{4}.$$

Gazeta Matematică

29. For any positive real numbers a, b, c show that the following inequality holds

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{c+a}{c+b} + \frac{a+b}{a+c} + \frac{b+c}{b+a}.$$

India, 2002

30. Let a, b, c be positive real numbers. Prove that

$$\frac{a^3}{b^2 - bc + c^2} + \frac{b^3}{c^2 - ac + a^2} + \frac{c^3}{a^2 - ab + b^2} \geq \frac{3(ab + bc + ca)}{a + b + c}.$$

Proposed for the Balkan Mathematical Olympiad

31. [Adrian Zahariuc] Consider the pairwise distinct integers $x_1, x_2, \ldots, x_n$, $n \geq 0$. Prove that

$$x_1^2 + x_2^2 + \cdots + x_n^2 \geq x_1 x_2 + x_2 x_3 + \cdots + x_n x_1 + 2n - 3.$$

32. [Murray Klamkin] Find the maximum value of the expression $x_1^2x_2+x_2^2x_3+\cdots+x_{n-1}^2x_n+x_n^2x_1$ when $x_1,x_2,\ldots,x_{n-1},x_n\geq 0$ add up to 1 and $n>2$.

Crux Mathematicorum

33. Find the maximum value of the constant c such that for any $x_1,x_2,\ldots,x_n,\cdots>0$ for which $x_{k+1}\geq x_1+x_2+\cdots+x_k$ for any k, the inequality

$$\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_n}\leq c\sqrt{x_1+x_2+\cdots+x_n}$$

also holds for any n.

IMO Shortlist, 1986

34. Given are positive real numbers a,b,c and x,y,z, for which $a+x=b+y=c+z=1$. Prove that

$$(abc+xyz)\Big(\frac{1}{ay}+\frac{1}{bz}+\frac{1}{cx}\Big)\geq 3.$$

Russia, 2002

35. [Viorel Vâjâitu, Alexandru Zaharescu] Let a,b,c be positive real numbers. Prove that

$$\frac{ab}{a+b+2c}+\frac{bc}{b+c+2a}+\frac{ca}{c+a+2b}\leq\frac{1}{4}(a+b+c).$$

Gazeta Matematică

36. Find the maximum value of the expression

$$a^3(b+c+d)+b^3(c+d+a)+c^3(d+a+b)+d^3(a+b+c)$$

where a,b,c,d are real numbers whose sum of squares is 1.

37. [Walther Janous] Let x,y,z be positive real numbers. Prove that

$$\frac{x}{x+\sqrt{(x+y)(x+z)}}+\frac{y}{y+\sqrt{(y+z)(y+x)}}+\frac{z}{z+\sqrt{(z+x)(z+y)}}\leq 1.$$

Crux Mathematicorum

38. Suppose that $a_1<a_2<\ldots<a_n$ are real numbers for some integer $n\geq 2$. Prove that

$$a_1a_2^4+a_2a_3^4+\ldots+a_na_1^4\geq a_2a_1^4+a_3a_2^4+\ldots+a_1a_n^4.$$

Iran, 1999

39. [Mircea Lascu] Let a,b,c be positive real numbers. Prove that

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\geq 4\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right).$$

40. Let $a_1, a_2, \ldots, a_n > 1$ be positive integers. Prove that at least one of the numbers $\sqrt[a_1]{a_2}, \sqrt[a_2]{a_3}, \ldots, \sqrt[a_{n-1}]{a_n}, \sqrt[a_n]{a_1}$ is less than or equal to $\sqrt[3]{3}$.

Adapted after a well-known problem

41. [Mircea Lascu, Marian Tetiva] Let x, y, z be positive real numbers which satisfy the condition

$$xy + xz + yz + 2xyz = 1.$$

Prove that the following inequalities hold

a) $xyz \leq \frac{1}{8}$;

b) $x + y + z \geq \frac{3}{2}$;

c) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 4(x + y + z)$;

d) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 4(x + y + z) \geq \frac{(2z-1)^2}{(z(2z+1))}$, where $z = \max\{x, y, z\}$.

42. [Manlio Marangelli] Prove that for any positive real numbers x, y, z,

$$3(x^2y + y^2z + z^2x)(xy^2 + yz^2 + zx^2) \geq xyz(x + y + z)^3.$$

43. [Gabriel Dospinescu] Prove that if a, b, c are real numbers such that $\max\{a, b, c\} - \min\{a, b, c\} \leq 1$, then

$$1 + a^3 + b^3 + c^3 + 6abc \geq 3a^2b + 3b^2c + 3c^2a$$

44. [Gabriel Dospinescu] Prove that for any positive real numbers a, b, c we have

$$27 + \left(2 + \frac{a^2}{bc}\right)\left(2 + \frac{b^2}{ca}\right)\left(2 + \frac{c^2}{ab}\right) \geq 6(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

45. Let $a_0 = \frac{1}{2}$ and $a_{k+1} = a_k + \frac{a_k^2}{n}$. Prove that $1 - \frac{1}{n} < a_n < 1$.

TST Singapore

46. [Călin Popa] Let a, b, c be positive real numbers, with $a, b, c \in (0, 1)$ such that $ab + bc + ca = 1$. Prove that

$$\frac{a}{1-a^2} + \frac{b}{1-b^2} + \frac{c}{1-c^2} \geq \frac{3}{4}\left(\frac{1-a^2}{a} + \frac{1-b^2}{b} + \frac{1-c^2}{c}\right).$$

47. [Titu Andreescu, Gabriel Dospinescu] Let $x, y, z \leq 1$ and $x + y + z = 1$. Prove that

$$\frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+z^2} \leq \frac{27}{10}.$$

48. [Gabriel Dospinescu] Prove that if $\sqrt{x}+\sqrt{y}+\sqrt{z}=1$, then

$$(1-x)^2(1-y)^2(1-z)^2 \geq 2^{15}xyz(x+y)(y+z)(z+x)$$

.

49. Let x, y, z be positive real numbers such that $xyz = x+y+z+2$. Prove that

(1) $xy+yz+zx \geq 2(x+y+z)$;

(2) $\sqrt{x}+\sqrt{y}+\sqrt{z} \leq \frac{3}{2}\sqrt{xyz}$.

50. Prove that if x, y, z are real numbers such that $x^2+y^2+z^2=2$, then

$$x+y+z \leq xyz+2.$$

IMO Shortlist, 1987

51. [Titu Andreescu, Gabriel Dospinescu] Prove that for any $x_1, x_2, \ldots, x_n \in (0,1)$ and for any permutation σ of the set $\{1,2,\ldots,n\}$, we have the inequality

$$\sum_{i=1}^{n} \frac{1}{1-x_i} \geq \left(1+\frac{\sum\limits_{i=1}^{n} x_i}{n}\right) \cdot \left(\sum_{i=1}^{n} \frac{1}{1-x_i \cdot x_{\sigma(i)}}\right).$$

52. Let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $\sum\limits_{i=1}^{n} \frac{1}{1+x_i} = 1$. Prove that

$$\sum_{i=1}^{n} \sqrt{x_i} \geq (n-1) \sum_{i=1}^{n} \frac{1}{\sqrt{x_i}}.$$

Vojtech Jarnik

53. [Titu Andreescu] Let $n > 3$ and $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1+a_2+\ldots+a_n \geq n$ and $a_1^2+a_2^2+\ldots+a_n^2 \geq n^2$. Prove that $max\{a_1, a_2, \ldots, a_n\} \geq 2$.

USAMO, 1999

54. [Vasile Cîrtoaje] If a, b, c, d are positive real numbers, then

$$\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+a}+\frac{d-a}{a+b} \geq 0.$$

55. If x and y are positive real numbers, show that $x^y+y^x > 1$.

France, 1996

56. Prove that if $a, b, c > 0$ have product 1, then

$$(a+b)(b+c)(c+a) \geq 4(a+b+c-1).$$

MOSP, 2001

57. Prove that for any $a, b, c > 0$,

$$(a^2+b^2+c^2)(a+b-c)(b+c-a)(c+a-b) \leq abc(ab+bc+ca).$$

58. [D.P.Mavlo] Let $a, b, c > 0$. Prove that

$$3+a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3\frac{(a+1)(b+1)(c+1)}{1+abc}.$$

Kvant, 1988

59. [Gabriel Dospinescu] Prove that for any positive real numbers $x_1, x_2, \ldots, x_n$ with product 1 we have the inequality

$$n^n \cdot \prod_{i=1}^{n} (x_i^n + 1) \geq \left(\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{1}{x_i}\right)^n.$$

60. Let $a, b, c, d > 0$ such that $a+b+c=1$. Prove that

$$a^3+b^3+c^3+abcd \geq \min\left\{\frac{1}{4}, \frac{1}{9}+\frac{d}{27}\right\}.$$

Kvant, 1993

61. Prove that for any real numbers a, b, c we have the inequality

$$\sum (1+a^2)^2(1+b^2)^2(a-c)^2(b-c)^2 \geq (1+a^2)(1+b^2)(1+c^2)(a-b)^2(b-c)^2(c-a)^2.$$

AMM

62. [Titu Andreescu, Mircea Lascu] Let α, x, y, z be positive real numbers such that $xyz = 1$ and $\alpha \geq 1$. Prove that

$$\frac{x^\alpha}{y+z}+\frac{y^\alpha}{z+x}+\frac{z^\alpha}{x+y} \geq \frac{3}{2}.$$

63. Prove that for any real numbers $x_1, \ldots, x_n, y_1, \ldots, y_n$ such that $x_1^2+\cdots+x_n^2 = y_1^2+\cdots+y_n^2 = 1$,

$$(x_1y_2-x_2y_1)^2 \leq 2\left(1-\sum_{k=1}^{n} x_k y_k\right).$$

Korea, 2001

64. [Laurenţiu Panaitopol] Let $a_1, a_2, \ldots, a_n$ be pairwise distinct positive integers. Prove that

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq \frac{2n+1}{3}(a_1 + a_2 + \cdots + a_n).$$

TST Romania

65. [Călin Popa] Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\frac{b\sqrt{c}}{a(\sqrt{3c} + \sqrt{ab})} + \frac{c\sqrt{a}}{b(\sqrt{3a} + \sqrt{bc})} + \frac{a\sqrt{b}}{c(\sqrt{3b} + \sqrt{ca})} \geq \frac{3\sqrt{3}}{4}.$$

66. [Titu Andreescu, Gabriel Dospinescu] Let a, b, c, d be real numbers such that $(1 + a^2)(1 + b^2)(1 + c^2)(1 + d^2) = 16$. Prove that

$$-3 \leq ab + bc + cd + da + ac + bd - abcd \leq 5.$$

67. Prove that

$$(a^2 + 2)(b^2 + 2)(c^2 + 2) \geq 9(ab + bc + ca)$$

for any positive real numbers a, b, c.

APMO, 2004

68. [Vasile Cîrtoaje] Prove that if $0 < x \leq y \leq z$ and $x + y + z = xyz + 2$, then
a) $(1 - xy)(1 - yz)(1 - xz) \geq 0$;
b) $x^2 y \leq 1, x^3 y^2 \leq \frac{32}{27}$.

69. [Titu Andreescu] Let a, b, c be positive real numbers such that $a+b+c \geq abc$. Prove that at least two of the inequalities

$$\frac{2}{a} + \frac{3}{b} + \frac{6}{c} \geq 6, \frac{2}{b} + \frac{3}{c} + \frac{6}{a} \geq 6, \frac{2}{c} + \frac{3}{a} + \frac{6}{b} \geq 6,$$

are true.

TST 2001, USA

70. [Gabriel Dospinescu, Marian Tetiva] Let $x, y, z > 0$ such that

$$x + y + z = xyz.$$

Prove that

$$(x - 1)(y - 1)(z - 1) \leq 6\sqrt{3} - 10.$$

71. [Marian Tetiva] Prove that for any positive real numbers a, b, c,

$$\left|\frac{a^3-b^3}{a+b}+\frac{b^3-c^3}{b+c}+\frac{c^3-a^3}{c+a}\right| \le \frac{(a-b)^2+(b-c)^2+(c-a)^2}{4}.$$

Moldova TST, 2004

72. [Titu Andreescu] Let a, b, c be positive real numbers. Prove that

$$(a^5-a^2+3)(b^5-b^2+3)(c^5-c^2+3) \ge (a+b+c)^3.$$

USAMO, 2004

73. [Gabriel Dospinescu] Let $n > 2$ and $x_1, x_2, \ldots, x_n > 0$ such that

$$\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} \frac{1}{x_k}\right) = n^2+1.$$

Prove that

$$\left(\sum_{k=1}^{n} x_k^2\right)\cdot\left(\sum_{k=1}^{n} \frac{1}{x_k^2}\right) > n^2+4+\frac{2}{n(n-1)}.$$

74. [Gabriel Dospinescu, Mircea Lascu, Marian Tetiva] Prove that for any positive real numbers a, b, c,

$$a^2+b^2+c^2+2abc+3 \ge (1+a)(1+b)(1+c).$$

75. [Titu Andreescu, Zuming Feng] Let a, b, c be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2}+\frac{(2b+a+c)^2}{2b^2+(a+c)^2}+\frac{(2c+a+b)^2}{2c^2+(a+b)^2} \le 8.$$

USAMO, 2003

76. Prove that for any positive real numbers x, y and any positive integers m, n,

$$(n-1)(m-1)(x^{m+n}+y^{m+n})+(m+n-1)(x^m y^n+x^n y^m) \ge mn(x^{m+n-1}y+y^{m+n-1}x).$$

Austrian-Polish Competition, 1995

77. Let a, b, c, d, e be positive real numbers such that $abcde = 1$. Prove that

$$\frac{a+abc}{1+ab+abcd}+\frac{b+bcd}{1+bc+bcde}+\frac{c+cde}{1+cd+cdea}+\frac{d+dea}{1+de+deab}+\frac{e+eab}{1+ea+eabc} \ge \frac{10}{3}.$$

Crux Mathematicorum

78. [Titu Andreescu] Prove that for any $a, b, c, \in \left(0, \frac{\pi}{2}\right)$ the following inequality holds

$$\frac{\sin a \cdot \sin(a-b) \cdot \sin(a-c)}{\sin(b+c)} + \frac{\sin b \cdot \sin(b-c) \cdot \sin(b-a)}{\sin(c+a)} + \frac{\sin c \cdot \sin(c-a) \cdot \sin(c-b)}{\sin(a+b)} \geq 0.$$

TST 2003, USA

79. Prove that if a, b, c are positive real numbers then,

$$\sqrt{a^4+b^4+c^4} + \sqrt{a^2b^2+b^2c^2+c^2a^2} \geq \sqrt{a^3b+b^3c+c^3a} + \sqrt{ab^3+bc^3+ca^3}.$$

KMO Summer Program Test, 2001

80. [Gabriel Dospinescu, Mircea Lascu] For a given $n > 2$ find the smallest constant k_n with the property: if $a_1, \ldots, a_n > 0$ have product 1, then

$$\frac{a_1a_2}{(a_1^2+a_2)(a_2^2+a_1)} + \frac{a_2a_3}{(a_2^2+a_3)(a_3^2+a_2)} + \cdots + \frac{a_na_1}{(a_n^2+a_1)(a_1^2+a_n)} \leq k_n.$$

81. [Vasile Cîrtoaje] For any real numbers a, b, c, x, y, z prove that the inequality holds

$$ax + by + cz + \sqrt{(a^2+b^2+c^2)(x^2+y^2+z^2)} \geq \frac{2}{3}(a+b+c)(x+y+z).$$

Kvant, 1989

82. [Vasile Cîrtoaje] Prove that the sides a, b, c of a triangle satisfy the inequality

$$3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-1\right) \geq 2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

83. [Walther Janous] Let $n > 2$ and let $x_1, x_2, \ldots, x_n > 0$ add up to 1. Prove that

$$\prod_{i=1}^{n}\left(1+\frac{1}{x_i}\right) \geq \prod_{i=1}^{n}\left(\frac{n-x_i}{1-x_i}\right).$$

Crux Mathematicorum

84. [Vasile Cîrtoaje, Gheorghe Eckstein] Consider positive real numbers $x_1, x_2, ..., x_n$ such that $x_1x_2...x_n = 1$. Prove that

$$\frac{1}{n-1+x_1} + \frac{1}{n-1+x_2} + ... + \frac{1}{n-1+x_n} \leq 1.$$

TST 1999, Romania

85. [Titu Andreescu] Prove that for any nonnegative real numbers a, b, c such that $a^2+b^2+c^2+abc = 4$ we have $0 \leq ab+bc+ca-abc \leq 2$.

USAMO, 2001

86. [Titu Andreescu] Prove that for any positive real numbers a, b, c the following inequality holds

$$\frac{a+b+c}{3} - \sqrt[3]{abc} \leq max\{(\sqrt{a}-\sqrt{b})^2, (\sqrt{b}-\sqrt{c})^2, (\sqrt{c}-\sqrt{a})^2\}.$$

TST 2000, USA

87. [Kiran Kedlaya] Let a,b,c be positive real numbers. Prove that

$$\frac{a+\sqrt{ab}+\sqrt[3]{abc}}{3} \leq \sqrt[3]{a \cdot \frac{a+b}{2} \cdot \frac{a+b+c}{3}}.$$

88. Find the greatest constant k such that for any positive integer n which is not a square, $|(1+\sqrt{n})\sin(\pi\sqrt{n})| > k$.

Vietnamese IMO Training Camp, 1995

89. [Dung Tran Nam] Let $x, y, z > 0$ such that $(x+y+z)^3 = 32xyz$. Find the minimum and maximum of $\dfrac{x^4+y^4+z^4}{(x+y+z)^4}$.

Vietnam, 2004

90. [George Tsintifas] Prove that for any $a, b, c, d > 0$,

$$(a+b)^3(b+c)^3(c+d)^3(d+a)^3 \geq 16a^2b^2c^2d^2(a+b+c+d)^4.$$

Crux Mathematicorum

91. [Titu Andreescu, Gabriel Dospinescu] Find the maximum value of the expression

$$\frac{(ab)^n}{1-ab} + \frac{(bc)^n}{1-bc} + \frac{(ca)^n}{1-ca}$$

where a, b, c are nonnegative real numbers which add up to 1 and n is some positive integer.

92. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \geq \frac{3}{\sqrt[3]{abc}(1+\sqrt[3]{abc})}.$$

93. [Dung Tran Nam] Prove that for any real numbers a, b, c such that $a^2+b^2+c^2 = 9$,

$$2(a+b+c) - abc \leq 10.$$

Vietnam, 2002

94. [Vasile Cîrtoaje] Let a, b, c be positive real numbers. Prove that

$$\left(a+\frac{1}{b}-1\right)\left(b+\frac{1}{c}-1\right)+\left(b+\frac{1}{c}-1\right)\left(c+\frac{1}{a}-1\right)+\left(c+\frac{1}{a}-1\right)\left(a+\frac{1}{b}-1\right)\geq 3.$$

95. [Gabriel Dospinescu] Let n be an integer greater than 2. Find the greatest real number m_n and the least real number M_n such that for any positive real numbers $x_1, x_2, \ldots, x_n$ (with $x_n = x_0, x_{n+1} = x_1$),

$$m_n \leq \sum_{i=1}^{n} \frac{x_i}{x_{i-1} + 2(n-1)x_i + x_{i+1}} \leq M_n.$$

96. [Vasile Cîrtoaje] If x, y, z are positive real numbers, then

$$\frac{1}{x^2+xy+y^2}+\frac{1}{y^2+yz+z^2}+\frac{1}{z^2+zx+z^2}\geq\frac{9}{(x+y+z)^2}.$$

Gazeta Matematică

97. [Vasile Cîrtoaje] For any $a, b, c, d > 0$ prove that

$$2(a^3+1)(b^3+1)(c^3+1)(d^3+1)\geq(1+abcd)(1+a^2)(1+b^2)(1+c^2)(1+d^2).$$

Gazeta Matematică

98. Prove that for any real numbers a, b, c,

$$(a+b)^4+(b+c)^4+(c+a)^4\geq\frac{4}{7}(a^4+b^4+c^4).$$

Vietnam TST, 1996

99. Prove that if a, b, c are positive real numbers such that $abc = 1$, then

$$\frac{1}{1+a+b}+\frac{1}{1+b+c}+\frac{1}{1+c+a}\leq\frac{1}{2+a}+\frac{1}{2+b}+\frac{1}{2+c}.$$

Bulgaria, 1997

100. [Dung Tran Nam] Find the minimum value of the expression $\frac{1}{a}+\frac{2}{b}+\frac{3}{c}$ where a, b, c are positive real numbers such that $21ab + 2bc + 8ca \leq 12$.

Vietnam, 2001

101. [Titu Andreescu, Gabriel Dospinescu] Prove that for any $x, y, z, a, b, c > 0$ such that $xy + yz + zx = 3$,

$$\frac{a}{b+c}(y+z)+\frac{b}{c+a}(z+x)+\frac{c}{a+b}(x+y)\geq 3.$$

102. Let a, b, c be positive real numbers. Prove that

$$\frac{(b+c-a)^2}{(b+c)^2+a^2}+\frac{(c+a-b)^2}{(c+a)^2+b^2}+\frac{(a+b-c)^2}{(a+b)^2+c^2}\geq\frac{3}{5}.$$

Japan, 1997

103. [Vasile Cîrtoaje, Gabriel Dospinescu] Prove that if $a_1, a_2, \ldots, a_n \geq 0$ then

$$a_1^n+a_2^n+\cdots+a_n^n-na_1a_2\ldots a_n\geq(n-1)\left(\frac{a_1+a_2+\cdots+a_{n-1}}{n-1}-a_n\right)^n$$

where a_n is the least among the numbers $a_1, a_2, \ldots, a_n$.

104. [Turkevici] Prove that for all positive real numbers x, y, z, t,

$$x^4+y^4+z^4+t^4+2xyzt\geq x^2y^2+y^2z^2+z^2t^2+t^2x^2+x^2z^2+y^2t^2.$$

Kvant

105. Prove that for any real numbers $a_1, a_2, \ldots, a_n$ the following inequality holds

$$\left(\sum_{i=1}^n a_i\right)^2\leq\sum_{i,j=1}^n\frac{ij}{i+j-1}a_ia_j.$$

106. Prove that if $a_1, a_2, \ldots, a_n, b_1, \ldots, b_n$ are real numbers between 1001 and 2002, inclusively, such that $a_1^2+a_2^2+\cdots+a_n^2=b_1^2+b_2^2+\cdots+b_n^2$, then we have the inequality

$$\frac{a_1^3}{b_1}+\frac{a_2^3}{b_2}+\cdots+\frac{a_n^3}{b_n}\leq\frac{17}{10}\left(a_1^2+a_2^2+\cdots+a_n^2\right).$$

TST Singapore

107. [Titu Andreescu, Gabriel Dospinescu] Prove that if a, b, c are positive real numbers which add up to 1, then

$$(a^2+b^2)(b^2+c^2)(c^2+a^2)\geq 8(a^2b^2+b^2c^2+c^2a^2)^2.$$

108. [Vasile Cîrtoaje] If a, b, c, d are positive real numbers such that $abcd = 1$, then

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}+\frac{1}{(1+c)^2}+\frac{1}{(1+d)^2}\geq 1.$$

Gazeta Matematică

109. [Vasile Cîrtoaje] Let a, b, c be positive real numbers. Prove that

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}\geq\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}.$$

Gazeta Matematică

110. [Gabriel Dospinescu] Let $a_1, a_2, \ldots, a_n$ be real numbers and let S be a non-empty subset of $\{1, 2, \ldots, n\}$. Prove that

$$\left(\sum_{i \in S} a_i\right)^2 \leq \sum_{1 \leq i \leq j \leq n} (a_i + \ldots + a_j)^2.$$

TST 2004, Romania

111. [Dung Tran Nam] Let $x_1, x_2 \ldots, x_{2004}$ be real numbers in the interval $[-1, 1]$ such that $x_1^3 + x_2^3 + \ldots + x_{2004}^3 = 0$. Find the maximal value of the $x_1 + x_2 + \cdots + x_{2004}$.

112. [Gabriel Dospinescu, Călin Popa] Prove that if $n \geq 2$ and $a_1, a_2, \ldots, a_n$ are real numbers with product 1, then

$$a_1^2 + a_2^2 + \cdots + a_n^2 - n \geq \frac{2n}{n-1} \cdot \sqrt[n]{n-1}(a_1 + a_2 + \cdots + a_n - n).$$

113. [Vasile Cîrtoaje] If a, b, c are positive real numbers, then

$$\sqrt{\frac{2a}{a+b}} + \sqrt{\frac{2b}{b+c}} + \sqrt{\frac{2c}{c+a}} \leq 3.$$

Gazeta Matematică

114. Prove the following inequality for positive real numbers x, y, z

$$(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq \frac{9}{4}$$

Iran, 1996

115. Prove that for any x, y in the interval $[0, 1]$,

$$\sqrt{1+x^2} + \sqrt{1+y^2} + \sqrt{(1-x)^2 + (1-y)^2} \geq (1 + \sqrt{5})(1 - xy).$$

116. [Suranyi] Prove that for any positive real numbers $a_1, a_2, \ldots, a_n$ the following inequality holds

$$(n-1)(a_1^n + a_2^n + \cdots + a_n^n) + n a_1 a_2 \ldots a_n \geq (a_1 + a_2 + \cdots + a_n)(a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1}).$$

Miklos Schweitzer Competition

117. Prove that for any $x_1, x_2, \ldots, x_n > 0$ with product 1,

$$\sum_{1 \leq i < j \leq n} (x_i - x_j)^2 \geq \sum_{i=1}^{n} x_i^2 - n.$$

A generalization of Turkevici's inequality

118. [Gabriel Dospinescu] Find the minimum value of the expression

$$\sum_{i=1}^{n}\sqrt{\frac{a_1a_2\ldots a_n}{1-(n-1)a_i}}$$

where $a_1, a_2, \ldots, a_n < \frac{1}{n-1}$ add up to 1 and $n > 2$ is an integer.

119. [Vasile Cîrtoaje] Let $a_1, a_2, \ldots, a_n < 1$ be nonnegative real numbers such that

$$a = \sqrt{\frac{a_1^2 + a_2^2 + \ldots + a_n^2}{n}} \geq \frac{\sqrt{3}}{3}.$$

Prove that

$$\frac{a_1}{1-a_1^2} + \frac{a_2}{1-a_2^2} + \ldots + \frac{a_n}{1-a_n^2} \geq \frac{na}{1-a^2}.$$

120. [Vasile Cîrtoaje, Mircea Lascu] Let a, b, c, x, y, z be positive real numbers such that

$$(a+b+c)(x+y+z) = (a^2+b^2+c^2)(x^2+y^2+z^2) = 4.$$

Prove that

$$abcxyz < \frac{1}{36}.$$

121. [Gabriel Dospinescu] For a given $n > 2$, find the minimal value of the constant k_n, such that if $x_1, x_2, \ldots, x_n > 0$ have product 1, then

$$\frac{1}{\sqrt{1+k_nx_1}} + \frac{1}{\sqrt{1+k_nx_2}} + \cdots + \frac{1}{\sqrt{1+k_nx_n}} \leq n-1.$$

Mathlinks Contest

122. [Vasile Cîrtoaje, Gabriel Dospinescu] For a given $n > 2$, find the maximal value of the constant k_n such that for any $x_1, x_2, \ldots, x_n > 0$ for which $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$ we have the inequality

$$(1-x_1)(1-x_2)\ldots(1-x_n) \geq k_nx_1x_2\ldots x_n.$$

CHAPTER 2

Solutions

1. Prove that the inequality

$$\sqrt{a^2+(1-b)^2}+\sqrt{b^2+(1-c)^2}+\sqrt{c^2+(1-a)^2}\geq \frac{3\sqrt{2}}{2}$$

holds for arbitrary real numbers a, b, c.

Kömal

***First solution*:**

Applying **Minkowsky's Inequality** to the left-hand side we have

$$\sqrt{a^2+(1-b)^2}+\sqrt{b^2+(1-c)^2}+\sqrt{c^2+(1-a)^2}\geq \sqrt{(a+b+c)^2+(3-a-b-c)^2}.$$

Denoting $a+b+c=x$ we get

$$(a+b+c)^2+(3-a-b-c)^2=2\left(x-\frac{3}{2}\right)^2+\frac{9}{2}\geq\frac{9}{2},$$

and the conclusion follows.

***Second solution*:**

We have the inequalities

$$\sqrt{a^2+(1-b)^2}+\sqrt{b^2+(1-c)^2}+\sqrt{c^2+(1-a)^2}\geq$$
$$\geq \frac{|a|+|1-b|}{\sqrt{2}}+\frac{|b|+|1-c|}{\sqrt{2}}+\frac{|c|+|1-a|}{\sqrt{2}}$$

and because $|x|+|1-x|\geq 1$ for all real numbers x the last quantity is at least $\frac{3\sqrt{2}}{2}$.

2. [Dinu Şerbănescu] If $a, b, c \in (0,1)$ prove that

$$\sqrt{abc}+\sqrt{(1-a)(1-b)(1-c)}<1.$$

Junior TST 2002, Romania

***First solution*:**

Observe that $x^{\frac{1}{2}} < x^{\frac{1}{3}}$ for $x\in(0,1)$. Thus

$$\sqrt{abc}<\sqrt[3]{abc},$$

and

$$\sqrt{(1-a)(1-b)(1-c)}<\sqrt[3]{(1-a)(1-b)(1-c)}.$$

By the **AM-GM Inequality**,

$$\sqrt{abc}<\sqrt[3]{abc}\leq\frac{a+b+c}{3},$$

and

$$\sqrt{(1-a)(1-b)(1-c)}<\sqrt[3]{(1-a)(1-b)(1-c)}\leq\frac{(1-a)+(1-b)+(1-c)}{3}.$$

Summing up, we obtain

$$\sqrt{abc}+\sqrt{(1-a)(1-b)(1-c)}<\frac{a+b+c+1-a+1-b+1-c}{3}=1,$$

as desired.

***Second solution*:**

We have

$$\sqrt{abc}+\sqrt{(1-a)(1-b)(1-c)}<\sqrt{b}\cdot\sqrt{c}+\sqrt{1-b}\cdot\sqrt{1-c}\leq 1,$$

by the **Cauchy-Schwarz Inequality**.

***Third solution*:**

Let $a=\sin^2 x, b=\sin^2 y, c=\sin^2 z$, where $x,y,z\in\left(0,\frac{\pi}{2}\right)$. The inequality becomes

$$\sin x\cdot\sin y\cdot\sin z+\cos x\cdot\cos y\cdot\cos z<1$$

and it follows from the inequalities

$$\sin x\cdot\sin y\cdot\sin z+\cos x\cdot\cos y\cdot\cos z<\sin x\cdot\sin y+\cos x\cdot\cos y=\cos(x-y)\leq 1$$

3. [Mircea Lascu] Let a,b,c be positive real numbers such that $abc=1$. Prove that

$$\frac{b+c}{\sqrt{a}}+\frac{c+a}{\sqrt{b}}+\frac{a+b}{\sqrt{c}}\geq\sqrt{a}+\sqrt{b}+\sqrt{c}+3.$$

Gazeta Matematică

***Solution*:**

From the **AM-GM Inequality**, we have

$$\frac{b+c}{\sqrt{a}}+\frac{c+a}{\sqrt{b}}+\frac{a+b}{\sqrt{c}}\geq 2\left(\sqrt{\frac{bc}{a}}+\sqrt{\frac{ca}{b}}+\sqrt{\frac{ab}{c}}\right)=$$

$$=\left(\sqrt{\frac{bc}{a}}+\sqrt{\frac{ca}{b}}\right)+\left(\sqrt{\frac{ca}{b}}+\sqrt{\frac{ab}{c}}\right)+\left(\sqrt{\frac{ab}{c}}+\sqrt{\frac{bc}{a}}\right)\geq$$

$$\geq 2(\sqrt{a}+\sqrt{b}+\sqrt{c})\geq\sqrt{a}+\sqrt{b}+\sqrt{c}+3\sqrt[6]{abc}=\sqrt{a}+\sqrt{b}+\sqrt{c}+3.$$

4. If the equation $x^4+ax^3+2x^2+bx+1=0$ has at least one real root, then $a^2+b^2\geq 8$.

Tournament of the Towns, 1993

Solution:

Let x be the real root of the equation. Using the **Cauchy-Schwarz Inequality** we infer that

$$a^2+b^2 \geq \frac{\left(x^4+2x^2+1\right)^2}{x^2+x^6} \geq 8,$$

because the last inequality is equivalent to $(x^2-1)^4 \geq 0$.

5. Find the maximum value of the expression $x^3+y^3+z^3-3xyz$ where $x^2+y^2+z^2=1$ and x, y, z are real numbers.

Solution:

Let $t = xy+yz+zx$. Let us observe that

$$(x^3+y^3+z^3-3xyz)^2 = (x+y+z)^2(1-xy-yz-zx)^2 = (1+2t)(1-t)^2.$$

We also have $-\frac{1}{2} \leq t \leq 1$. Thus, we must find the maximum value of the expression $(1+2t)(1-t)^2$, where $-\frac{1}{2} \leq t \leq 1$. In this case we clearly have $(1+2t)(t-1)^2 \leq 1 \Leftrightarrow t^2(3-2t) \geq 0$ and thus $|x^3+y^3+z^3-3xyz| \leq 1$. We have equality for $x=1,\ y=z=0$ and thus the maximum value is 1.

6. Let a, b, c, x, y, z be positive real numbers such that $x+y+z=1$. Prove that

$$ax+by+cz+2\sqrt{(xy+yz+zx)(ab+bc+ca)} \leq a+b+c.$$

Ukraine, 2001

First solution:

We will use the **Cauchy-Schwarz Inequality** twice. First, we can write $ax+by+cz \leq \sqrt{a^2+b^2+c^2} \cdot \sqrt{x^2+y^2+z^2}$ and then we apply again the **Cauchy-Schwarz Inequality** to obtain:

$$\begin{aligned} ax+by+cz &+ 2\sqrt{(xy+yz+zx)(ab+bc+ca)} \leq \\ &\leq \sqrt{\sum a^2} \cdot \sqrt{\sum x^2} + \sqrt{2\sum ab} \cdot \sqrt{2\sum xy} \leq \\ &\leq \sqrt{\sum x^2 + 2\sum xy} \cdot \sqrt{\sum a^2 + 2\sum ab} = \sum a. \end{aligned}$$

Second solution:

The inequality being homogeneous in a, b, c we can assume that $a+b+c=1$. We apply this time the **AM-GM Inequality** and we find that

$$ax+by+cz+2\sqrt{(xy+yz+zx)(ab+bc+ca)} \leq ax+by+cz+xy+yz+zx+ab+bc+ca.$$

Consequently,

$$xy+yz+zx+ab+bc+ca = \frac{1-x^2-y^2-z^2}{2} + \frac{1-a^2-b^2-c^2}{2} \leq 1-ax-by-cz,$$

the last one being equivalent to $(x-a)^2+(y-b)^2+(z-c)^2 \geq 0$.

7. [Darij Grinberg] If a, b, c are three positive real numbers, then

$$\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2} \geq \frac{9}{4(a+b+c)}.$$

First solution:

We rewrite the inequality as

$$(a+b+c)\left(\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2}\right) \geq \frac{9}{4}.$$

Applying the **Cauchy-Schwarz Inequality** we get

$$(a+b+c)\left(\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2}\right) \geq \left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^2.$$

It remains to prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2}$$

which is well-known.

Second solution:

Without loss of generality we may assume that $a+b+c=1$. Now, consider the function $f:(0,1)\to(0,\infty)$, $f(x)=\dfrac{x}{(1-x)^2}$. A short computation of derivatives shows that f is convex. Thus, we may apply **Jensen's Inequality** and the conclusion follows.

8. [Hojoo Lee] Let $a,b,c \geq 0$. Prove that

$$\sqrt{a^4+a^2b^2+b^4}+\sqrt{b^4+b^2c^2+c^4}+\sqrt{c^4+c^2a^2+a^4} \geq$$
$$\geq a\sqrt{2a^2+bc}+b\sqrt{2b^2+ca}+c\sqrt{2c^2+ab}.$$

Gazeta Matematică

Solution:

We start from $(a^2-b^2)^2 \geq 0$. We rewrite it as $4a^4+4a^2b^2+4b^4 \geq 3a^4+6a^2b^2+3b^4$. It follows that $\sqrt{a^4+a^2b^2+b^2} \geq \frac{\sqrt{3}}{2}(a^2+b^2)$.

Using this observation, we find that

$$\left(\sum\sqrt{a^4+a^2b^2+b^4}\right)^2 \geq 3\left(\sum a^2\right)^2.$$

But using the **Cauchy-Schwarz Inequality** we obtain

$$\left(\sum a\sqrt{2a^2+bc}\right)^2 \leq \left(\sum a^2\right)\left(\sum\left(2a^2+bc\right)\right) \leq 3\left(\sum a^2\right)^2$$

and the inequality is proved.

9. If a, b, c are positive real numbers such that $abc = 2$, then

$$a^3 + b^3 + c^3 \geq a\sqrt{b+c} + b\sqrt{c+a} + c\sqrt{a+b}.$$

When does equality hold?

JBMO 2002 Shortlist

***First solution*:**

Applying the **Cauchy-Schwarz Inequality** gives

$$3(a^2 + b^2 + c^2) \geq 3(a + b + c)^2 \qquad (1)$$

and

$$(a^2 + b^2 + c^2)^2 \leq (a + b + c)(a^3 + b^3 + c^3). \qquad (2)$$

These two inequalities combined yield

$$\begin{aligned} a^3 + b^3 + c^3 &\geq \frac{(a^2 + b^2 + c^2)(a + b + c)}{3} \\ &= \frac{(a^2 + b^2 + c^2)[(b + c) + (a + c) + (a + b)]}{6} \\ &\geq \frac{(a\sqrt{b+c} + b\sqrt{a+c} + c\sqrt{a+b})^2}{6} \qquad (3) \end{aligned}$$

Using the **AM-GM Inequality** we obtain

$$\begin{aligned} a\sqrt{b+c} + b\sqrt{a+c} + c\sqrt{a+b} &\geq 3\sqrt[3]{abc\left(\sqrt{(a+b)(b+c)(c+a)}\right)} \\ &\geq 3\sqrt[3]{abc\sqrt{8abc}} = 3\sqrt[3]{8} = 6. \end{aligned}$$

Thus

$$(a\sqrt{b+c} + b\sqrt{a+c} + c\sqrt{a+b})^2 \geq 6(a\sqrt{b+c} + b\sqrt{a+c} + c\sqrt{a+b}). \qquad (4)$$

The desired inequality follows from (3) and (4).

***Second solution*:**

We have

$$a\sqrt{b+c} + b\sqrt{c+a} + c\sqrt{a+b} \leq \sqrt{2(a^2 + b^2 + c^2)(a + b + c)}.$$

Using **Chebyshev Inequality**, we infer that

$$\sqrt{2(a^2 + b^2 + c^2)(a + b + c)} \leq \sqrt{6(a^3 + b^3 + c^3)}$$

and so it is enough to prove that $a^3 + b^3 + c^3 \geq 3abc$, which is true by the **AM-GM Inequality**. We have equality if $a = b = c = \sqrt[3]{2}$.

10. [Ioan Tomescu] Let $x, y, z > 0$. Prove that

$$\frac{xyz}{(1 + 3x)(x + 8y)(y + 9z)(z + 6)} \leq \frac{1}{7^4}.$$

When do we have equality?

Gazeta Matematică

Solution:

First, we write the inequality in the following form

$$(1+3x)\left(1+\frac{8y}{x}\right)\left(1+\frac{9z}{y}\right)\left(1+\frac{6}{z}\right)\geq 7^4.$$

But this follows immediately from **Huygens Inequality**. We have equality for $x=2, y=\frac{3}{2}, z=1$.

11. [Mihai Piticari, Dan Popescu] Prove that

$$5(a^2+b^2+c^2)\leq 6(a^3+b^3+c^3)+1,$$

for all $a,b,c>0$ with $a+b+c=1$.

Solution:

Because $a+b+c=1$, we have $a^3+b^3+c^3=3abc+a^2+b^2+c^2-ab-bc-ca$. The inequality becomes

$$\begin{aligned}
5(a^2+b^2+c^2) &\leq 18abc+6(a^2+b^2+c^2)-6(ab+bc+ca)+1\Leftrightarrow\\
&\Leftrightarrow 18abc+1-2(ab+bc+ca)+1\geq 6(ab+bc+ca)\ \Leftrightarrow\\
&\Leftrightarrow 8(ab+bc+ca)\leq 2+18abc\Leftrightarrow 4(ab+bc+ca)\leq 1+9abc\ \Leftrightarrow\\
&\Leftrightarrow (1-2a)(1-2b)(1-2c)\leq abc\Leftrightarrow\\
&\Leftrightarrow (b+c-a)(c+a-b)(a+b-c)\leq abc,
\end{aligned}$$

which is equivalent to **Schur's Inequality**.

12. [Mircea Lascu] Let $x_1,x_2,\ldots,x_n\in R$, $n\geq 2$ and $a>0$ such that $x_1+x_2+\ldots+x_n=a$ and $x_1^2+x_2^2+\ldots+x_n^2\leq\frac{a^2}{n-1}$. Prove that $x_i\in\left[0,\frac{2a}{n}\right]$, for all $i\in\{1,2,\ldots,n\}$.

Solution:

Using the **Cauchy-Schwarz Inequality**, we get

$$(a-x_1)^2\leq(n-1)\left(x_2^2+x_3^2+\cdots+x_n^2\right)\leq(n-1)\left(\frac{a^2}{n-1}-x_1^2\right).$$

Thus,

$$a^2-2ax_1+x_1^2\leq a^2-(n-1)x_1^2\ \Leftrightarrow\ x_1\left(x_1-\frac{2a}{n}\right)\leq 0$$

and the conclusion follows.

13. [Adrian Zahariuc] Prove that for any $a, b, c \in (1,2)$ the following inequality holds

$$\frac{b\sqrt{a}}{4b\sqrt{c}-c\sqrt{a}}+\frac{c\sqrt{b}}{4c\sqrt{a}-a\sqrt{b}}+\frac{a\sqrt{c}}{4a\sqrt{b}-b\sqrt{c}}\geq 1.$$

Solution:

The fact that $a, b, c \in (1,2)$ makes all denominators positive. Then

$$\frac{b\sqrt{a}}{4b\sqrt{c}-c\sqrt{a}}\geq\frac{a}{a+b+c} \quad\Leftrightarrow\quad b(a+b+c)\geq\sqrt{a}(4b\sqrt{c}-c\sqrt{a}) \quad\Leftrightarrow$$

$$\Leftrightarrow\quad (a+b)(b+c)\geq 4b\sqrt{ac},$$

the last one coming from $a+b\geq 2\sqrt{ab}$ and $b+c\geq 2\sqrt{bc}$. Writing the other two inequalities and adding them up give the desired result.

14. For positive real numbers a, b, c such that $abc \leq 1$, prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq a+b+c.$$

First solution:

If $ab+bc+ca\leq a+b+c$, then the **Cauchy-Schwarz Inequality** solves the problem:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=\frac{a^2c+b^2a+c^2b}{abc}\geq\frac{\dfrac{(a+b+c)^2}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}}{abc}\geq a+b+c.$$

Otherwise, the same inequality gives

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=\frac{a^2c+b^2a+c^2b}{abc}\geq\frac{\dfrac{(ab+bc+ca)^2}{a+b+c}}{abc}\geq a+b+c$$

(here we have used the fact that $abc\leq 1$).

Second solution:

Replacing a, b, c by ta, tb, tc with $t=\dfrac{1}{\sqrt[3]{abc}}$ preserves the value of the quantity in the left-hand side of the inequality and increases the value of the right-hand side and makes $at\cdot bt\cdot ct=abct^3=1$. Hence we may assume without loss of generality that $abc=1$. Then there exist positive real numbers x, y, z such that $a=\dfrac{y}{x}, b=\dfrac{z}{y}, c=\dfrac{x}{z}$. The **Rearrangement Inequality** gives

$$x^3+y^3+z^3\geq x^2y+y^2z+z^2x.$$

Thus
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=\frac{x^3+y^3+z^3}{xyz}\geq\frac{x^2y+y^2z+z^2x}{xyz}=a+b+c$$
as desired.

***Third solution*:**

Using the **AM-GM Inequality**, we deduce that
$$\frac{2a}{b}+\frac{b}{c}\geq 3\sqrt[3]{\frac{a^2}{bc}}\geq 3a.$$
Similarly, $\frac{2b}{c}+\frac{c}{a}\geq 3b$ and $\frac{2c}{a}+\frac{a}{b}\geq 3c$. Adding these three inequalities, the conclusion is immediate.

***Forth solution*:**

Let $x=\sqrt[9]{\frac{ab^4}{c^2}}, y=\sqrt[9]{\frac{ca^4}{b^2}}, z=\sqrt[9]{\frac{bc^4}{a^2}}$. Consequently, $a=xy^2, b=zx^2, c=yz^2$, and also $xyz\leq 1$. Thus, using the **Rearrangement Inequality**, we find that
$$\sum\frac{a}{b}=\sum\frac{x^2}{yz}\geq xyz\sum\frac{x^2}{yz}=\sum x^3\geq\sum xy^2=\sum a.$$

15. [Vasile Cîrtoaje, Mircea Lascu] Let a,b,c,x,y,z be positive real numbers such that $a+x\geq b+y\geq c+z$ and $a+b+c=x+y+z$. Prove that $ay+bx\geq ac+xz$.

***Solution*:**

We have
$$ay+bx-ac-xz=a(y-c)+x(b-z)=a(a+b-x-z)+x(b-z)=a(a-x)+(a+x)(b-z)=$$
$$=\frac{1}{2}(a-x)^2+\frac{1}{2}(a^2-x^2)+(a+x)(b-z)=\frac{1}{2}(a-x)^2+\frac{1}{2}(a+x)(a-x+2b-2z)=$$
$$=\frac{1}{2}(a-x)^2+\frac{1}{2}(a+x)(b-c+y-z)\geq 0.$$
The equality occurs when $a=x$, $b=z$, $c=y$ and $2x\geq y+z$.

16. [Vasile Cîrtoaje, Mircea Lascu] Let a,b,c be positive real numbers so that $abc=1$. Prove that
$$1+\frac{3}{a+b+c}\geq\frac{6}{ab+ac+bc}.$$
Junior TST 2003, Romania

***Solution*:**

We set $x=\frac{1}{a}$, $y=\frac{1}{b}$, $z=\frac{1}{c}$ and observe that $xyz=1$. The inequality is equivalent to
$$1+\frac{3}{xy+yz+zx}\geq\frac{6}{x+y+z}.$$

From $(x+y+z)^2 \geq 3(xy+yz+zx)$ we get

$$1+\frac{3}{xy+yz+zx} \geq 1+\frac{9}{(x+y+z)^2},$$

so it suffices to prove that

$$1+\frac{9}{(x+y+z)^2} \geq \frac{6}{x+y+z}.$$

The last inequality is equivalent to $\left(1-\frac{3}{x+y+z}\right)^2 \geq 0$ and this ends the proof.

17. Let a, b, c be positive real numbers. Prove that

$$\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2} \geq \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}.$$

JBMO 2002 Shortlist

***First solution*:**

We have

$$\frac{a^3}{b^2} \geq \frac{a^2}{b}+a-b \Leftrightarrow a^3+b^3 \geq ab(a+b) \Leftrightarrow (a-b)^2(a+b) \geq 0,$$

which is clearly true. Writing the analogous inequalities and adding them up gives

$$\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2} \geq \frac{a^2}{b}+a-b+\frac{b^2}{c}+b-c+\frac{c^2}{a}+c-a=\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}.$$

***Second solution*:**

By the **Cauchy-Schwarz Inequality** we have

$$(a+b+c)\left(\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2}\right) \geq \left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)^2,$$

so we only have to prove that

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \geq a+b+c.$$

But this follows immediately from the **Cauchy-Schwarz Inequality**.

18. Prove that if $n>3$ and $x_1, x_2, \ldots, x_n>0$ have product 1, then

$$\frac{1}{1+x_1+x_1x_2}+\frac{1}{1+x_2+x_2x_3}+\cdots+\frac{1}{1+x_n+x_nx_1}>1.$$

Russia, 2004

***Solution*:**

We use a similar form of the classical substitution $x_1=\frac{a_2}{a_1}, x_2=\frac{a_3}{a_2}, \ldots, x_n=\frac{a_1}{a_n}$. In this case the inequality becomes

$$\frac{a_1}{a_1+a_2+a_3}+\frac{a_2}{a_2+a_3+a_4}+\cdots+\frac{a_n}{a_n+a_1+a_2}>1$$

and it is clear, because $n > 3$ and $a_i + a_{i+1} + a_{i+2} < a_1 + a_2 + \cdots + a_n$ for all i.

19. [Marian Tetiva] Let x, y, z be positive real numbers satisfying the condition

$$x^2 + y^2 + z^2 + 2xyz = 1.$$

Prove that

a) $xyz \le \frac{1}{8}$;

b) $x + y + z \le \frac{3}{2}$;

c) $xy + xz + yz \le \frac{3}{4} \le x^2 + y^2 + z^2$;

d) $xy + xz + yz \le \frac{1}{2} + 2xyz$.

Solution:

a) This is very simple. From the **AM-GM Inequality**, we have

$$1 = x^2 + y^2 + z^2 + 2xyz \ge 4\sqrt[4]{2x^3y^3z^3} \Rightarrow x^3y^3z^3 \le \frac{1}{2 \cdot 4^4} \Rightarrow xyz \le \frac{1}{8}.$$

b) Clearly, we must have $x, y, z \in (0, 1)$. If we put $s = x+y+z$, we get immediately from the given relation

$$s^2 - 2s + 1 = 2(1-x)(1-y)(1-z).$$

Then, again by the **AM-GM Inequality** ($1-x$, $1-y$, $1-z$ being positive), we obtain

$$s^2 - 2s + 1 \le 2\left(\frac{1-x+1-y+1-z}{3}\right)^3 = 2\left(\frac{3-s}{3}\right)^3.$$

After some easy calculations this yields

$$2s^3 + 9s^2 - 27 \le 0 \Leftrightarrow (2s-3)(s+3)^2 \le 0$$

and the conclusion is plain.

c) These inequalities are simple consequences of a) and b):

$$xy + xz + yz \le \frac{(x+y+z)^2}{3} \le \frac{1}{3} \cdot \frac{9}{4} = \frac{3}{4}$$

and

$$x^2 + y^2 + z^2 = 1 - 2xyz \ge 1 - 2 \cdot \frac{1}{8} = \frac{3}{4}.$$

d) This is more delicate; we first notice that there are always two of the three numbers, both greater (or both less) than $\frac{1}{2}$. Because of symmetry, we may assume that $x, y \le \frac{1}{2}$, or $x, y \ge \frac{1}{2}$ and then

$$(2x-1)(2y-1) \ge 0 \Leftrightarrow x + y - 2xy \le \frac{1}{2}.$$

On the other hand,

$$\begin{aligned} 1 &= x^2 + y^2 + z^2 + 2xyz \geq 2xy + z^2 + 2xyz \Rightarrow \\ &\Rightarrow 2xy\left(1+z\right) \leq 1 - z^2 \Rightarrow 2xy \leq 1 - z. \end{aligned}$$

Now, we only have to multiply side by side the inequalities from above

$$x + y - 2xy \leq \frac{1}{2}$$

and

$$z \leq 1 - 2xy$$

to get the desired result:

$$xz + yz - 2xyz \leq \frac{1}{2} - xy \Leftrightarrow xy + xz + yz \leq \frac{1}{2} + 2xyz.$$

It is not important in the proof, but also notice that

$$x + y - 2xy = xy\left(\frac{1}{x} + \frac{1}{y} - 2\right) > 0,$$

because $\frac{1}{x}$ and $\frac{1}{y}$ are both greater than 1.

Remark.

1) One can obtain some other inequalities, using

$$z + 2xy \leq 1$$

and the two likes

$$y + 2xz \leq 1,\ x + 2yz \leq 1.$$

For example, multiplying these inequalities by z, y, x respectively and adding the new inequalities, we get

$$x^2 + y^2 + z^2 + 6xyz \leq x + y + z,$$

or

$$1 + 4xyz \leq x + y + z.$$

2) If ABC is any triangle, the numbers

$$x = \sin\frac{A}{2},\ y = \sin\frac{B}{2},\ z = \sin\frac{C}{2}$$

satisfy the condition of this problem; conversely, if $x, y, z > 0$ verify

$$x^2 + y^2 + z^2 + 2xyz = 1$$

then there is a triangle ABC so that

$$x = \sin\frac{A}{2},\ y = \sin\frac{B}{2},\ z = \sin\frac{C}{2}.$$

According to this, new proofs can be given for such inequalities.

20. [Marius Olteanu] Let $x_1, x_2, x_3, x_4, x_5 \in R$ so that $x_1+x_2+x_3+x_4+x_5 = 0$. Prove that

$$|\cos x_1| + |\cos x_2| + |\cos x_3| + |\cos x_4| + |\cos x_5| \geq 1.$$

Gazeta Matematică

Solution:

It is immediate to prove that

$$|\sin(x+y)| \leq \min\{|\cos x| + |\cos y|, |\sin x| + |\sin y|\}$$

and

$$|\cos(x+y)| \leq \min\{|\sin x| + |\cos y|, |\sin y| + |\cos x|\}.$$

Thus, we infer that

$$1 = \left|\cos\left(\sum_{i=1}^{5} x_i\right)\right| \leq |\cos x_1| + \left|\sin\left(\sum_{i=2}^{5} x_i\right)\right| \leq |\cos x_1| + |\cos x_2| + |\cos(x_3+x_4+x_5)|.$$

But in the same manner we can prove that

$$|\cos(x_3+x_4+x_5)| \leq |\cos x_3| + |\cos x_4| + |\cos x_5|$$

and the conclusion follows.

21. [Florina Cârlan, Marian Tetiva] Prove that if $x, y, z > 0$ satisfy the condition

$$x + y + z = xyz$$

then $xy + xz + yz \geq 3 + \sqrt{x^2+1} + \sqrt{y^2+1} + \sqrt{z^2+1}$.

First solution:

We have

$$xyz = x + y + z \geq 2\sqrt{xy} + z \Rightarrow z\left(\sqrt{xy}\right)^2 - 2\sqrt{xy} - z \geq 0.$$

Because the positive root of the trinomial $zt^2 - 2t - z$ is

$$\frac{1+\sqrt{1+z^2}}{z},$$

we get from here

$$\sqrt{xy} \geq \frac{1+\sqrt{1+z^2}}{z} \Leftrightarrow z\sqrt{xy} \geq 1 + \sqrt{1+z^2}.$$

Of course, we have two other similar inequalities. Then,

$$\begin{aligned} xy + xz + yz &\geq x\sqrt{yz} + y\sqrt{xz} + z\sqrt{xy} \geq \\ &\geq 3 + \sqrt{x^2+1} + \sqrt{y^2+1} + \sqrt{z^2+1}, \end{aligned}$$

and we have both a proof of the given inequality, and a little improvement of it.

***Second solution*:**

Another improvement is as follows. Start from

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\geq\frac{1}{xy}+\frac{1}{xz}+\frac{1}{yz}=1\Rightarrow x^2y^2+x^2z^2+y^2z^2\geq x^2y^2z^2,$$

which is equivalent to

$$(xy+xz+yz)^2\geq 2xyz\,(x+y+z)+x^2y^2z^2=3\,(x+y+z)^2\,.$$

Further on,

$$\begin{aligned}&(xy+xz+yz-3)^2=(xy+xz+yz)^2-6\,(xy+xz+yz)+9\geq\\&\geq 3\,(x+y+z)^2-6\,(xy+xz+yz)+9=3\left(x^2+y^2+z^2\right)+9,\end{aligned}$$

so that

$$xy+xz+yz\geq 3+\sqrt{3\,(x^2+y^2+z^2)+9}.$$

But

$$\sqrt{3\,(x^2+y^2+z^2)+9}\geq\sqrt{x^2+1}+\sqrt{y^2+1}+\sqrt{z^2+1}$$

is a consequence of the **Cauchy-Schwarz Inequality** and we have a second improvement and proof for the desired inequality:

$$\begin{aligned}xy+xz+yz &\geq 3+\sqrt{3\,(x^2+y^2+z^2)+9}\geq\\&\geq 3+\sqrt{x^2+1}+\sqrt{y^2+1}+\sqrt{z^2+1}.\end{aligned}$$

22. [Laurenţiu Panaitopol] Prove that

$$\frac{1+x^2}{1+y+z^2}+\frac{1+y^2}{1+z+x^2}+\frac{1+z^2}{1+x+y^2}\geq 2,$$

for any real numbers $x,y,z>-1$.

JBMO, 2003

***Solution*:**

Let us observe that $y\leq\dfrac{1+y^2}{2}$ and $1+y+z^2>0$, so

$$\frac{1+x^2}{1+y+z^2}\geq\frac{1+x^2}{1+z^2+\dfrac{1+y^2}{2}}$$

and the similar relations. Setting $a=1+x^2$, $b=1+y^2$, $c=1+z^2$, it is sufficient to prove that

$$\frac{a}{2c+b}+\frac{b}{2a+c}+\frac{c}{2b+a}\geq 1 \qquad (1)$$

for any $a, b, c > 0$. Let $A = 2c + b$, $B = 2a + c$, $C = 2b + a$. Then $a = \dfrac{C + 4B - 2A}{9}$, $b = \dfrac{A + 4C - 2B}{9}$, $c = \dfrac{B + 4A - 2C}{9}$ and the inequality (1) is rewritten as

$$\frac{C}{A} + \frac{A}{B} + \frac{B}{C} + 4\left(\frac{B}{A} + \frac{C}{B} + \frac{A}{C}\right) \geq 15.$$

Because $A, B, C > 0$, we have from the **AM-GM Inequality** that

$$\frac{C}{A} + \frac{A}{B} + \frac{B}{C} \geq 3\sqrt[3]{\frac{A}{B} \cdot \frac{B}{C} \cdot \frac{C}{A}} = 3$$

and $\dfrac{B}{A} + \dfrac{C}{B} + \dfrac{A}{C} \geq 3$ and the conclusion follows.

An alternative solution for (1) is by using the **Cauchy-Schwarz Inequality**:

$$\frac{a}{2c+b} + \frac{b}{2a+c} + \frac{c}{2b+a} = \frac{a^2}{2ac+ab} + \frac{b^2}{2ab+cb} + \frac{c^2}{2bc+ac} \geq \frac{(a+b+c)^2}{3(ab+bc+ca)} \geq 1.$$

23. Let $a, b, c > 0$ with $a + b + c = 1$. Show that

$$\frac{a^2+b}{b+c} + \frac{b^2+c}{c+a} + \frac{c^2+a}{a+b} \geq 2.$$

Solution:

Using the **Cauchy-Schwarz Inequality**, we find that

$$\sum \frac{a^2+b}{b+c} \geq \frac{\left(\sum a^2 + 1\right)^2}{\sum a^2(b+c) + \sum a^2 + \sum ab}.$$

And so it is enough to prove that

$$\frac{\left(\sum a^2 + 1\right)^2}{\sum a^2(b+c) + \sum a^2 + \sum ab} \geq 2 \ \Leftrightarrow \ 1 + \left(\sum a^2\right)^2 \geq 2\sum a^2(b+c) + 2\sum ab.$$

The last inequality can be transformed as follows

$$\begin{aligned} 1 + \left(\sum a^2\right)^2 &\geq 2\sum a^2(b+c) + 2\sum ab \Leftrightarrow 1 + \left(\sum a^2\right)^2 \geq \\ &\geq 2\sum a^2 - 2\sum a^3 + 2\sum ab \Leftrightarrow \left(\sum a^2\right)^2 + 2\sum a^3 \geq \sum a^2, \end{aligned}$$

and it is true, because

$$\sum a^3 \geq \frac{\sum a^2}{3} \ \textbf{(Chebyshev's Inequality)}$$

and

$$\left(\sum a^2\right)^2 \geq \frac{\sum a^2}{3}.$$

24. Let $a, b, c \geq 0$ such that $a^4 + b^4 + c^4 \leq 2(a^2b^2 + b^2c^2 + c^2a^2)$. Prove that

$$a^2 + b^2 + c^2 \leq 2(ab + bc + ca).$$

Kvant, 1988

***Solution*:**

The condition

$$\sum a^4 \leq 2\sum a^2b^2$$

is equivalent to

$$(a+b+c)(a+b-c)(b+c-a)(c+a-b) \geq 0.$$

In any of the cases $a = b + c,\ b = c + a,\ c = a + b$, the inequality

$$\sum a^2 \leq 2\sum ab$$

is clear. So, suppose $a + b \neq c,\ b + c \neq a,\ c + a \neq b$. Because at most one of the numbers $b + c - a,\ c + a - b,\ a + b - c$ is negative and their product is nonnegative, all of them are positive. Thus, we may assume that

$$a^2 < ab + ac,\ b^2 < bc + ba,\ c^2 < ca + cb$$

and the conclusion follows.

25. Let $n \geq 2$ and $x_1, ..., x_n$ be positive real numbers satisfying

$$\frac{1}{x_1 + 1998} + \frac{1}{x_2 + 1998} + ... + \frac{1}{x_n + 1998} = \frac{1}{1998}.$$

Prove that

$$\frac{\sqrt[n]{x_1 x_2 ... x_n}}{n-1} \geq 1998.$$

Vietnam, 1998

***Solution*:**

Let $\dfrac{1998}{1998 + x_i} = a_i$. The problem reduces to proving that for any positive real numbers $a_1, a_2, \ldots, a_n$ such that $a_1 + a_2 + \cdots + a_n = 1$ we have the inequality

$$\prod_{i=1}^{n} \left(\frac{1}{a_i} - 1\right) \geq (n-1)^n.$$

This inequality can be obtained by multiplying the inequalities

$$\begin{aligned} \frac{1}{a_i} - 1 &= \frac{a_1 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_n}{a_i} \geq \\ &\geq (n-1)\sqrt[n-1]{\frac{a_1 \ldots a_{i-1} a_{i+1} \ldots a_n}{a_i^{n-1}}} \end{aligned}$$

26. [Marian Tetiva] Consider positive real numbers x, y, z so that

$$x^2 + y^2 + z^2 = xyz.$$

Prove the following inequalities

a) $xyz \geq 27$;

b) $xy + xz + yz \geq 27$;

c) $x + y + z \geq 9$;

d) $xy + xz + yz \geq 2(x + y + z) + 9$.

Solution:

a), b), c) Using well-known inequalities, we have

$$xyz = x^2 + y^2 + z^2 \geq 3\sqrt[3]{x^2y^2z^2} \Rightarrow (xyz)^3 \geq 27(xyz)^2,$$

which yields $xyz \geq 27$. Then

$$xy + xz + yz \geq 3\sqrt[3]{(xyz)^2} \geq 3\sqrt[3]{27^2} = 27$$

and

$$x + y + z \geq 3\sqrt[3]{xyz} \geq 3\sqrt[3]{27} = 9.$$

d) We notice that $x^2 < xyz \Rightarrow x < yz$ and the likes. Consequently,

$$xy < yz \cdot xz \Rightarrow 1 < z^2 \Rightarrow z > 1.$$

Hence all the three numbers must be greater than 1. Set

$$a = x - 1,\ b = y - 1,\ c = z - 1.$$

We then have $a > 0, b > 0, c > 0$ and

$$x = a + 1,\ y = b + 1,\ z = c + 1.$$

Replacing these in the given condition we get

$$(a+1)^2 + (b+1)^2 + (c+1)^2 = (a+1)(b+1)(c+1)$$

which is the same as

$$a^2 + b^2 + c^2 + a + b + c + 2 = abc + ab + ac + bc.$$

If we put $q = ab + ac + bc$, we have

$$q \leq a^2 + b^2 + c^2,\ \sqrt{3q} \leq a + b + c$$

and

$$abc \leq \left(\frac{q}{3}\right)^{\frac{3}{2}} = \frac{(3q)^{\frac{3}{2}}}{27}.$$

Thus

$$\begin{aligned} q+\sqrt{3q}+2 &\le a^2+b^2+c^2+a+b+c+2= \\ &= abc+ab+ac+bc \le \frac{\left(\sqrt{3q}\right)^3}{27}+q \end{aligned}$$

and setting $x=\sqrt{3q}$, we have

$$x+2 \le \frac{x^3}{27} \Leftrightarrow (x-6)(x+3)^2 \ge 0.$$

Finally,

$$\sqrt{3q}=x\ge 6 \Rightarrow q=ab+ac+bc\ge 12.$$

Now, recall that $a=x-1$, $b=y-1$, $c=z-1$, so we get

$$\begin{aligned} (x-1)(y-1) &+ (x-1)(z-1)+(y-1)(z-1)\ge 12 \Rightarrow \\ &\Rightarrow xy+xz+yz\ge 2(x+y+z)+9; \end{aligned}$$

and we are done.

One can also prove the stronger inequality

$$xy+xz+yz\ge 4(x+y+z)-9.$$

Try it!

27. Let x,y,z be positive real numbers with sum 3. Prove that

$$\sqrt{x}+\sqrt{y}+\sqrt{z}\ge xy+yz+zx.$$

Russia, 2002

***Solution*:**

Rewrite the inequality in the form

$$x^2+2\sqrt{x}+y^2+2\sqrt{y}+z^2+2\sqrt{z}\ge x^2+y^2+z^2+2xy+2yz+2zx \Leftrightarrow$$

$$\Leftrightarrow x^2+2\sqrt{x}+y^2+2\sqrt{y}+z^2+2\sqrt{z}\ge 9.$$

Now, from the **AM-GM Inequality**, we have

$$x^2+2\sqrt{x}=x^2+\sqrt{x}+\sqrt{x}\ge 3\sqrt[3]{x^2\cdot x}=3x,$$

$$y^2+2\sqrt{y}\ge 3y,\ z^2+2\sqrt{z}\ge 3z,$$

hence

$$x^2+y^2+z^2+2\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)\ge 3(x+y+z)\ge 9.$$

28. [D. Olteanu] Let a,b,c be positive real numbers. Prove that

$$\frac{a+b}{b+c}\cdot\frac{a}{2a+b+c}+\frac{b+c}{c+a}\cdot\frac{b}{2b+c+a}+\frac{c+a}{a+b}\cdot\frac{c}{2c+a+b}\ge\frac{3}{4}.$$

Gazeta Matematică

Solution:

We set $x = a + b$, $y = b + c$ and $z = a + c$ and after a few computations we obtain the equivalent form

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} + \frac{x}{x+y} + \frac{y}{y+z} + \frac{z}{z+x} \geq \frac{9}{2}.$$

But using the **Cauchy-Schwarz Inequality**,

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} + \frac{x}{x+y} + \frac{y}{y+z} + \frac{z}{z+x} \geq \frac{(x+y+z)^2}{xy+yz+zx} + \frac{(x+y+z)^2}{xy+yz+zx+x^2+y^2+z^2} =$$

$$\frac{2(x+y+z)^4}{2(xy+yz+zx)(xy+yz+zx+x^2+y^2+z^2)} \geq \frac{8(x+y+z)^4}{(xy+yz+zx+(x+y+z)^2)^2}$$

and the conclusion follows.

29. For any positive real numbers a, b, c show that the following inequality holds

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{c+a}{c+b} + \frac{a+b}{a+c} + \frac{b+c}{b+a}.$$

India, 2002

Solution:

Let us take $\frac{a}{b} = x, \frac{b}{c} = y, \frac{c}{a} = z$. Observe that

$$\frac{a+c}{b+c} = \frac{1+xy}{1+y} = x + \frac{1-x}{1+y}.$$

Using similar relations, the problem reduces to proving that if $xyz = 1$, then

$$\frac{x-1}{y+1} + \frac{y-1}{z+1} + \frac{z-1}{x+1} \geq 0 \Leftrightarrow$$

$$\Leftrightarrow (x^2-1)(z+1) + (y^2-1)(x+1) + (z^2-1)(y+1) \geq 0 \Leftrightarrow$$

$$\Leftrightarrow \sum x^2 z + \sum x^2 \geq \sum x + 3.$$

But this inequality is very easy. Indeed, using the **AM-GM Inequality** we have $\sum x^2 z \geq 3$ and so it remains to prove that $\sum x^2 \geq \sum x$, which follows from the inequalities

$$\sum x^2 \geq \frac{\left(\sum x\right)^2}{3} \geq \sum x.$$

30. Let a, b, c be positive real numbers. Prove that

$$\frac{a^3}{b^2 - bc + c^2} + \frac{b^3}{c^2 - ac + a^2} + \frac{c^3}{a^2 - ab + b^2} \geq \frac{3(ab+bc+ca)}{a+b+c}.$$

Proposed for the Balkan Mathematical Olympiad

First solution:

Since $a+b+c \geq \dfrac{3(ab+bc+ca)}{a+b+c}$, it suffices to prove that:

$$\frac{a^3}{b^2-bc+c^2}+\frac{b^3}{c^2-ca+a^2}+\frac{c^3}{a^2-ab+b^2} \geq a+b+c.$$

From the **Cauchy-Schwartz Inequality**, we get

$$\sum \frac{a^3}{b^2-bc+c^2} \geq \frac{\left(\sum a^2\right)^2}{\sum a(b^2-bc+c^2)}.$$

Thus we have to show that

$$(a^2+b^2+c^2)^2 \geq (a+b+c) \cdot \sum a(b^2-bc+c^2).$$

This inequality is equivalent to

$$a^4+b^4+c^4+abc(a+b+c) \geq ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2),$$

which is just **Schur's Inequality**.

Remark.

The inequality

$$\frac{a^3}{b^2-bc+c^2}+\frac{b^3}{c^2-ca+a^2}+\frac{c^3}{a^2-ab+b^2} \geq a+b+c$$

was proposed by Vasile Cârtoaje in **Gazeta Matematică** as a special case ($n=3$) of the more general inequality

$$\frac{2a^n-b^n-c^n}{b^2-bc+c^2}+\frac{2b^n-c^n-a^n}{c^2-ca+a^2}+\frac{2c^n-a^n-b^n}{a^2-ab+b^2} \geq 0.$$

Second solution:

Rewrite our inequality as

$$\sum \frac{(b+c)a^3}{b^3+c^3} \geq \frac{3(ab+bc+ca)}{a+b+c}.$$

But this follows from a more general result:

If $a, b, c, x, y, z > 0$ then

$$\sum \frac{a(y+z)}{b+c} \geq 3\frac{\sum xy}{\sum x}.$$

But this inequality is an immediate (and weaker) consequence of the result from problem 101.

Third solution:

Let

$$A = \sum \frac{a^3}{b^2-bc+c^2}$$

and

$$B=\sum \frac{b^3+c^3}{b^2-bc+c^2}=\sum \frac{(b+c)(b^2-bc+c^2)}{b^2-bc+c^2}=2\sum a.$$

So we get

$$\begin{aligned} A+B &= \left(\sum a^3\right)\left(\sum \frac{1}{b^2-bc+c^2}\right)= \\ &= \frac{1}{2}\left(\sum (b+c)(b^2-bc+c^2)\right)\left(\sum \frac{1}{b^2-bc+c^2}\right)\geq \frac{1}{2}\left(\sum \sqrt{b+c}\right)^2, \end{aligned}$$

from the **Cauchy-Schwarz Inequality**. Hence

$$A\geq \frac{1}{2}\left(\sum \sqrt{b+c}\right)^2-2\sum a=\sum \sqrt{a+b}\cdot\sqrt{b+c}-\sum a.$$

We denote

$$A_a=\sqrt{c+a}\cdot\sqrt{b+a}-a=\frac{\sum ab}{\sqrt{\sum ab+a^2}+a}$$

and the similar relations with A_b and A_c. So $A\geq A_a+A_b+A_c$. But because $\left(\sum a\right)^2\geq 3\left(\sum ab\right)$ we get

$$A_a\geq \frac{\sum ab}{\sqrt{\frac{\left(\sum a\right)^2}{3}+a^2}+a}=\frac{3\sum ab}{\left(\sum a\right)^2}\left(\sqrt{\frac{\left(\sum a\right)^2}{3}+a^2}-a\right)$$

and also the similar inequalities are true. So we only need to prove that

$$\sum\left(\sqrt{\frac{\left(\sum a\right)^2}{3}+a^2}-a\right)\geq a+b+c \;\Leftrightarrow\; \sum\sqrt{\frac{1}{3}+\left(\frac{a}{a+b+c}\right)^2}\geq 2.$$

We consider the convex function $f(t)=\sqrt{\frac{1}{3}+t^2}$. Using **Jensen's Inequality** we finally deduce that

$$\sum\sqrt{\frac{1}{3}+\left(\frac{a}{a+b+c}\right)^2}\geq 2.$$

We have equality if and only if $a=b=c$.

31. [Adrian Zahariuc] Consider the pairwise distinct integers $x_1,x_2,\ldots,x_n$, $n\geq 0$. Prove that

$$x_1^2+x_2^2+\cdots+x_n^2\geq x_1x_2+x_2x_3+\cdots+x_nx_1+2n-3.$$

Solution:

The inequality can be rewritten as

$$\sum_{i=1}^{n}(x_i - x_{i+1})^2 \geq 2(2n-3).$$

Let $x_m = \min\{x_1, x_2, \ldots, x_n\}$ and $x_M = \max\{x_1, x_2, \ldots, x_n\}$. Suppose without loss of generality that $m < M$. Let

$$S_1 = (x_m - x_{m+1})^2 + \cdots + (x_{M-1} - x_M)^2$$

and

$$S_2 = (x_M - x_{M+1})^2 + \cdots + (x_n - x_1)^2 + (x_1 - x_2)^2 + \cdots + (x_{m-1} - x_m)^2.$$

The inequality $\sum_{i=1}^{k} a_i^2 \geq \frac{1}{k}\left(\sum_{i=1}^{k} a_i\right)^2$ (which follows from the **Cauchy-Schwarz Inequality**) implies that

$$S_1 \geq \frac{(x_M - x_m)^2}{M-m}$$

and

$$S_2 \geq \frac{(x_M - x_m)^2}{n-(M-m)}.$$

So

$$\begin{aligned}\sum_{i=1}^{n}(x_i - x_{i+1})^2 &= S_1 + S_2 \geq (x_M - x_m)^2\left(\frac{1}{M-m} + \frac{1}{n-(M-m)}\right) \geq \\ &\geq (n-1)^2\frac{4}{n} = 4n - 8 + \frac{4}{n} > 4n-8.\end{aligned}$$

But

$$\sum_{i=1}^{n}(x_i - x_{i+1})^2 \equiv \sum_{i=1}^{n} x_i - x_{i+1} = 0 \bmod 2$$

so

$$\sum_{i=1}^{n}(x_i - x_{i+1})^2 \geq 4n-6$$

and the problem is solved.

32. [Murray Klamkin] Find the maximum value of the expression $x_1^2x_2 + x_2^2x_3 + \cdots + x_{n-1}^2x_n + x_n^2x_1$ when $x_1, x_2, \ldots, x_{n-1}, x_n \geq 0$ add up to 1 and $n > 2$.

Crux Mathematicorum

Solution:

First of all, it is clear that the maximum is at least $\frac{4}{27}$, because this value is attained for $x_1 = \frac{1}{3}, x_2 = \frac{2}{3}, x_3 = \cdots = x_n = 0$. Now, we will prove by induction that

$$x_1^2x_2 + x_2^2x_3 + \cdots + x_{n-1}^2x_n + x_n^2x_1 \leq \frac{4}{27}$$

for all $x_1, x_2, \ldots, x_{n-1}, x_n \geq 0$ which add up to 1. Let us prove first the inductive step. Suppose the inequality is true for n and we will prove it for $n+1$. We can of course assume that $x_2 = \min\{x_1, x_2, \ldots, x_{n+1}\}$. But this implies that

$$x_1^2x_2 + x_2^2x_3 + \cdots + x_n^2x_1 \leq (x_1+x_2)^2x_3 + x_3^2x_4 + \cdots + x_{n-1}^2x_n + x_n^2(x_1+x_2).$$

But from the inductive hypothesis we have

$$(x_1+x_2)^2x_3 + x_3^2x_4 + \cdots + x_{n-1}^2x_n + x_n^2(x_1+x_2) \leq \frac{4}{27}$$

and the inductive step is proved. Thus, it remains to prove that $a^2b+b^2c+c^2a \leq \frac{4}{27}$ if $a+b+c=1$. We may of course assume that a is the greatest among a, b, c. In this case the inequality $a^2b+b^2c+c^2a \leq \left(a+\frac{c}{2}\right)^2 \cdot \left(b+\frac{c}{2}\right)$ follows immediately from $abc \geq b^2c, \frac{a^2c}{2} \geq \frac{ac^2}{2}$. Because

$$1 = \frac{a+\frac{c}{2}}{2} + \frac{a+\frac{c}{2}}{2} + b + \frac{c}{2} \geq 3\sqrt[3]{\frac{\left(b+\frac{c}{2}\right)\left(a+\frac{c}{2}\right)^2}{4}},$$

we have proved that $a^2b+b^2c+c^2a \leq \frac{4}{27}$ and this shows that the maximum is indeed $\frac{4}{27}$.

33. Find the maximum value of the constant c such that for any $x_1, x_2, \ldots, x_n, \cdots > 0$ for which $x_{k+1} \geq x_1 + x_2 + \cdots + x_k$ for any k, the inequality

$$\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n} \leq c\sqrt{x_1+x_2+\cdots+x_n}$$

also holds for any n.

IMO Shortlist, 1986

Solution:

First, let us see what happens if x_{k+1} and $x_1+x_2+\cdots+x_k$ are close for any k. For example, we can take $x_k = 2^k$, because in this case we have $x_1+x_2+\cdots+x_k = x_{k+1}-2$. Thus, we find that

$$c \geq \frac{\sum_{k=1}^{n} \sqrt{2}^k}{\sqrt{\sum_{k=1}^{n} 2^k}}$$

for any n. Taking the limit, we find that $c \geq 1+\sqrt{2}$. Now, let us prove that $1+\sqrt{2}$ works. We will prove the inequality

$$\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n} \leq (1+\sqrt{2})\sqrt{x_1+x_2+\cdots+x_n}$$

by induction. For $n = 1$ or $n = 2$ it is clear. Suppose it is true for n and we will prove that $\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n} + \sqrt{x_{n+1}} \leq (1+\sqrt{2})\sqrt{x_1 + x_2 + \cdots + x_n + x_{n+1}}$. Of course, it is enough to prove that

$$\sqrt{x_{n+1}} \leq (1+\sqrt{2})\left(\sqrt{x_1 + x_2 + \cdots + x_{n+1}} - \sqrt{x_1 + x_2 + \cdots + x_n}\right)$$

which is equivalent to

$$\sqrt{x_1 + x_2 + \cdots + x_n} + \sqrt{x_1 + x_2 + \cdots + x_{n+1}} \leq (1+\sqrt{2})\sqrt{x_{n+1}}.$$

But this one follows because

$$x_1 + x_2 + \cdots + x_n \leq x_{n+1}.$$

34. Given are positive real numbers a, b, c and x, y, z, for which $a + x = b + y = c + z = 1$. Prove that

$$(abc + xyz)\left(\frac{1}{ay} + \frac{1}{bz} + \frac{1}{cx}\right) \geq 3.$$

Russia, 2002

Solution:

Let us observe that $abc + xyz = (1-b)(1-c) + ac + ab - a$. Thus,

$$\frac{1-c}{a} + \frac{c}{1-b} - 1 = \frac{abc + xyz}{a(1-b)}.$$

Using these identities we deduce immediately that

$$3 + (xyz + abc)\left(\frac{1}{ay} + \frac{1}{bz} + \frac{1}{cx}\right) = \frac{a}{1-c} + \frac{b}{1-a} + \frac{c}{1-b} + \frac{1-c}{a} + \frac{1-a}{b} + \frac{1-b}{c}.$$

Now, all we have to do is apply the **AM-GM Inequality**

$$\frac{a}{1-c} + \frac{b}{1-a} + \frac{c}{1-b} + \frac{1-c}{a} + \frac{1-a}{b} + \frac{1-b}{c} \geq 6.$$

35. [Viorel Vâjâitu, Alexandru Zaharescu] Let a, b, c be positive real numbers. Prove that

$$\frac{ab}{a+b+2c} + \frac{bc}{b+c+2a} + \frac{ca}{c+a+2b} \leq \frac{1}{4}(a+b+c).$$

Gazeta Matematică

First solution:

We have the following chain of inequalities

$$\sum \frac{ab}{a+b+2c} = \sum \frac{ab}{a+c+b+c} \leq \sum \frac{ab}{4}\left(\frac{1}{a+c} + \frac{1}{b+c}\right) = \frac{a+b+c}{4}.$$

Second solution:

Because the inequality is homogeneous we can consider without loss of generality that $a+b+c=1$ and so the inequality is equivalent to

$$\sum \frac{1}{a(a+1)} \leq \frac{1}{4abc}.$$

We have $\dfrac{1}{t(t+1)} = \dfrac{1}{t} - \dfrac{1}{t+1}$, so the inequality is equivalent to

$$\sum \frac{1}{a} \leq \sum \frac{1}{a+1} + \frac{1}{4abc}.$$

We will prove now the following intercalation:

$$\sum \frac{1}{a} \leq \frac{9}{4} + \frac{1}{4abc} \leq \sum \frac{1}{a+1} + \frac{1}{4abc}.$$

The inequality in the right follows from the **Cauchy-Schwarz Inequality**:

$$\left(\sum \frac{1}{a+1}\right)\left(\sum (a+1)\right) \geq 9$$

and the identity $\sum (a+1) = 4$. The inequality in the left can be written as $\sum ab \leq \dfrac{1+9abc}{4}$, which is exactly **Schur's Inequality**.

36. Find the maximum value of the expression

$$a^3(b+c+d) + b^3(c+d+a) + c^3(d+a+b) + d^3(a+b+c)$$

where a, b, c, d are real numbers whose sum of squares is 1.

Solution:

The idea is to observe that $a^3(b+c+d)+b^3(c+d+a)+c^3(d+a+b)+d^3(a+b+c)$ is equal to $\sum ab(a^2+b^2)$. Now, because the expression $ab(a^2+b^2)$ appears when writing $(a-b)^4$, let us see how the initial expression can be written:

$$\begin{aligned} \sum ab(a^2+b^2) &= \sum \frac{a^4+b^4+6a^2b^2-(a-b)^4}{4} = \\ &= \frac{3\sum a^4 + 6\sum a^2b^2 - \sum (a-b)^4}{4} = \frac{3-\sum (a-b)^4}{4} \leq \frac{3}{4}. \end{aligned}$$

The maximum is attained for $a=b=c=d=\dfrac{1}{2}$.

37. [Walther Janous] Let x, y, z be positive real numbers. Prove that

$$\frac{x}{x+\sqrt{(x+y)(x+z)}} + \frac{y}{y+\sqrt{(y+z)(y+x)}} + \frac{z}{z+\sqrt{(z+x)(z+y)}} \leq 1.$$

Crux Mathematicorum

First solution:

We have $(x+y)(x+z) = xy+(x^2+yz)+xz \geq xy+2x\sqrt{yz}+xz = (\sqrt{xy}+\sqrt{xz})^2$.
Hence

$$\sum \frac{x}{x+\sqrt{(x+y)(x+z)}} \leq \sum \frac{x}{x+\sqrt{xy}+\sqrt{xz}}.$$

But

$$\sum \frac{x}{x+\sqrt{xy}+\sqrt{xz}} = \sum \frac{\sqrt{x}}{\sqrt{x}+\sqrt{y}+\sqrt{z}} = 1$$

and this solves the problem.

Second solution:

From **Huygens Inequality** we have $\sqrt{(x+y)(x+z)} \geq x+\sqrt{yz}$ and using this inequality for the similar ones we get

$$\frac{x}{x+\sqrt{(x+y)(x+z)}} + \frac{y}{y+\sqrt{(y+z)(y+x)}} + \frac{z}{z+\sqrt{(z+x)(z+y)}} \leq$$
$$\leq \frac{x}{2x+\sqrt{yz}} + \frac{y}{2y+\sqrt{zx}} + \frac{z}{2z+\sqrt{xy}}.$$

Now, we denote with $a = \dfrac{\sqrt{yz}}{x}$, $b = \dfrac{\sqrt{zx}}{y}$, $c = \dfrac{\sqrt{xy}}{z}$ and the inequality becomes

$$\frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c} \leq 1.$$

From the above notations we can see that $abc = 1$, so the last inequality becomes after clearing the denominators $ab+bc+ca \geq 3$, which follows from the **AM-GM Inequality**.

38. Suppose that $a_1 < a_2 < ... < a_n$ are real numbers for some integer $n \geq 2$. Prove that

$$a_1a_2^4 + a_2a_3^4 + ... + a_na_1^4 \geq a_2a_1^4 + a_3a_2^4 + ... + a_1a_n^4.$$

Iran, 1999

Solution:

A quick look shows that as soon as we prove the inequality for $n = 3$, it will be proved by induction for larger n. Thus, we must prove that for any $a < b < c$ we have $ab(b^3-a^3)+bc(c^3-b^3) \geq ca(c^3-a^3) \Leftrightarrow (c^3-b^3)(ac-bc) \leq (b^3-a^3)(ab-ac)$. Because $a < b < c$, the last inequality reduces to $a(b^2+ab+a^2) \leq c(c^2+bc+b^2)$. And this last inequality is equivalent to $(c-a)(a^2+b^2+c^2+ab+bc+ca) \geq 0$, which is clear.

39. [Mircea Lascu] Let a, b, c be positive real numbers. Prove that

$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 4\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right).$$

Solution:
Using the inequality $\dfrac{1}{x+y} \leq \dfrac{1}{4x} + \dfrac{1}{4y}$ we infer that

$$\frac{4a}{b+c} \leq \frac{a}{b} + \frac{a}{c}, \quad \frac{4b}{a+c} \leq \frac{b}{a} + \frac{b}{c} \text{ and } \frac{4c}{a+b} \leq \frac{c}{a} + \frac{c}{b}.$$

Adding up these three inequalities, the conclusion follows.

40. Let $a_1, a_2, \ldots, a_n > 1$ be positive integers. Prove that at least one of the numbers $\sqrt[a_1]{a_2}, \sqrt[a_2]{a_3}, \ldots, \sqrt[a_{n-1}]{a_n}, \sqrt[a_n]{a_1}$ is less than or equal to $\sqrt[3]{3}$.

Adapted after a well-known problem

Solution:
Suppose we have $a_{i+1}^{\frac{1}{a_i}} > 3^{\frac{1}{3}}$ for all i. First, we will prove that $n^{\frac{1}{n}} \leq 3^{\frac{1}{3}}$ for all natural number n. For $n = 1, 2, 3, 4$ it is clear. Suppose the inequality is true for $n > 3$ and let us prove it for $n+1$. This follows from the fact that

$$1 + \frac{1}{n} \leq 1 + \frac{1}{4} < \sqrt[3]{3} \Rightarrow 3^{\frac{n+1}{3}} = \sqrt[3]{3} \cdot 3^{\frac{n}{3}} \geq \frac{n+1}{n} \cdot n = n+1.$$

Thus, using this observation, we find that $a_{i+1}^{\frac{1}{a_i}} > 3^{\frac{1}{3}} \geq a_{i+1}^{\frac{1}{a_{i+1}}} \Rightarrow a_{i+1} > a_i$ for all i, which means that $a_1 < a_2 < \cdots < a_{n-1} < a_n < a_1$, contradiction.

41. [Mircea Lascu, Marian Tetiva] Let x, y, z be positive real numbers which satisfy the condition

$$xy + xz + yz + 2xyz = 1.$$

Prove that the following inequalities hold

a) $xyz \leq \dfrac{1}{8}$;

b) $x + y + z \geq \dfrac{3}{2}$;

c) $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} \geq 4(x+y+z)$;

d) $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} - 4(x+y+z) \geq \dfrac{(2z-1)^2}{(z(2z+1))}$, where $z = \max\{x, y, z\}$.

Solution:
a) We put $t^3 = xyz$; according to the **AM-GM Inequality** we have

$$1 = xy + xz + yz + 2xyz \geq 3t^2 + 2t^3 \Leftrightarrow (2t-1)(t+1)^2 \leq 0,$$

therefore $2t - 1 \leq 0 \Leftrightarrow t \leq \dfrac{1}{2}$, this meaning that $xyz \leq \dfrac{1}{8}$.

b) Denote also $s = x + y + z$; the following inequalities are well-known

$$(x+y+z)^2 \geq 3(xy + xz + yz)$$

and

$$(x+y+z)^3 \geq 27xyz;$$

then we have $2s^3 \geq 54xyz = 27 - 27(xy + xz + yz) \geq 27 - 9s^2$, i. e.

$$2s^3 + 9s^2 - 27 \geq 0 \Leftrightarrow (2s-3)(s+3)^2 \geq 0,$$

where from $2s - 3 \geq 0 \Leftrightarrow s \geq \frac{3}{2}$.

Or, because $p \leq \frac{1}{8}$, we have

$$s^2 \geq 3q = 3(1-2p) \geq 3\left(1 - \frac{2}{8}\right) = \frac{9}{4};$$

if we put $q = xy + xz + yz$, $p = xyz$.

Now, one can see the following is also true

$$q = xy + xz + yz \geq \frac{3}{4}.$$

c) The three numbers x, y, z cannot be all less than $\frac{1}{2}$, because, in this case we get the contradiction

$$xy + xz + yz + 2xyz < \frac{3}{4} + 2 \cdot \frac{1}{8} = 1;$$

because of symmetry we may assume then that $z \geq \frac{1}{2}$.

We have $1 = (2z+1)xy + z(x+y) \geq (2z+1)xy + 2z\sqrt{xy}$, which can also be written in the form $\left((2z+1)\sqrt{xy} - 1\right)\left(\sqrt{xy} + 1\right) \leq 0$; and this one yields the inequality

$$xy \leq \frac{1}{(2z+1)^2}.$$

We also have $1 = (2z+1)xy + z(x+y) \leq (2z+1)\frac{(x+y)^2}{4} + z(x+y)$, consequently $((2z+1)(x+y) - 2)(x+y+2) \geq 0$, which shows us that

$$x + y \geq \frac{2}{2z+1}.$$

The inequality to be proved

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 4(x+y+z)$$

can be also rearranged as

$$(x+y)\left(\frac{1}{xy} - 4\right) \geq \frac{4z^2 - 1}{z} = \frac{(2z-1)(2z+1)}{z}.$$

From the above calculations we infer that

$$(x+y)\left(\frac{1}{xy} - 4\right) \geq \frac{2}{2z+1}\left((2z+1)^2 - 4\right) = \frac{2(2z-1)(2z+3)}{2z+1}$$

(the assumption $z \geq \frac{1}{2}$ allows the multiplication of the inequalities side by side), and this means that the problem would be solved if we proved

$$\frac{2(2z+3)}{2z+1} \geq \frac{2z+1}{z} \Leftrightarrow 4z^2+6z \geq 4z^2+4z+1;$$

but this follows for $z \geq \frac{1}{2}$ and we are done.

d) Of course, if z is the greatest from the numbers x, y, z, then $z \geq \frac{1}{2}$; we saw that

$$\begin{aligned} \frac{1}{x}+\frac{1}{y}-4(x+y) &= (x+y)\left(\frac{1}{xy}-4\right) \geq \frac{2}{2z+1}(2z+1) \\ &= \frac{2(2z-1)(2z+3)}{2z+1} = 4z-\frac{1}{z}+\frac{(2z-1)^2}{z(2z+1)}, \end{aligned}$$

where from we get our last inequality

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-4(x+y+z) \geq \frac{(2z-1)^2}{z(2z+1)}.$$

Of course, in the right-hand side z could be replaced by any of the three numbers which is $\geq \frac{1}{2}$ (two such numbers could be, surely there is one).

Remark.

It is easy to see that the given condition implies the existence of positive numbers a, b, c such that $x = \frac{a}{b+c}, y = \frac{b}{c+a}, z = \frac{c}{a+b}$. And now *a*), *b*) and *c*) reduce immediately to well-known inequalities! Try to prove using this substitution *d*).

42. [Manlio Marangelli] Prove that for any positive real numbers x, y, z,

$$3(x^2y+y^2z+z^2x)(xy^2+yz^2+zx^2) \geq xyz(x+y+z)^3.$$

***Solution*:**

Using the **AM-GM Inequality**, we find that

$$\frac{1}{3}+\frac{y^2z}{y^2z+z^2x+x^2y}+\frac{xy^2}{yz^2+zx^2+xy^2} \geq \frac{3y\sqrt[3]{xyz}}{\sqrt[3]{3\cdot(y^2z+z^2x+x^2y)\cdot(yz^2+zx^2+xy^2)}}$$

and two other similar relations

$$\frac{1}{3}+\frac{z^2x}{y^2z+z^2x+x^2y}+\frac{yz^2}{yz^2+zx^2+xy^2} \geq \frac{3z\sqrt[3]{xyz}}{\sqrt[3]{3\cdot(y^2z+z^2x+x^2y)\cdot(yz^2+zx^2+xy^2)}}$$

$$\frac{1}{3}+\frac{x^2y}{y^2z+z^2x+x^2y}+\frac{zx^2}{yz^2+zx^2+xy^2} \geq \frac{3x\sqrt[3]{xyz}}{\sqrt[3]{3\cdot(y^2z+z^2x+x^2y)\cdot(yz^2+zx^2+xy^2)}}$$

Then, adding up the three relations, we find exactly the desired inequality.

43. [Gabriel Dospinescu] Prove that if a, b, c are real numbers such that $\max\{a,b,c\} - \min\{a,b,c\} \leq 1$, then

$$1 + a^3 + b^3 + c^3 + 6abc \geq 3a^2b + 3b^2c + 3c^2a$$

***Solution*:**

Clearly, we may assume that $a = \min\{a,b,c\}$ and let us write $b = a + x, c = a + y$, where $x, y \in [0,1]$. It is easy to see that $a^3 + b^3 + c^3 - 3abc = 3a(x^2 - xy + y^2) + x^3 + y^3$ and $a^2b + b^2c + c^2a - 3abc = a(x^2 - xy + y^2) + x^2y$. So, the inequality becomes $1 + x^3 + y^3 \geq 3x^2y$. But this follows from the fact that $1 + x^3 + y^3 \geq 3xy \geq 3x^2y$, because $0 \leq x, y \leq 1$.

44. [Gabriel Dospinescu] Prove that for any positive real numbers a, b, c we have

$$27 + \left(2 + \frac{a^2}{bc}\right)\left(2 + \frac{b^2}{ca}\right)\left(2 + \frac{c^2}{ab}\right) \geq 6(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

***Solution*:**

By expanding the two sides, the inequality is equivalent to $2abc(a^3 + b^3 + c^3 + 3abc - a^2b - a^2c - b^2a - b^2c - c^2a - c^2b) + (a^3b^3 + b^3c^3 + c^3a^3 + 3a^2b^2c^2 - a^3b^2c - a^3bc^2 - ab^3c^2 - ab^2c^3 - a^2b^3c - a^2bc^3) \geq 0$. But this is true from **Schur's Inequality** applied for a, b, c and ab, bc, ca.

45. Let $a_0 = \dfrac{1}{2}$ and $a_{k+1} = a_k + \dfrac{a_k^2}{n}$. Prove that $1 - \dfrac{1}{n} < a_n < 1$.

TST Singapore

***Solution*:**

We have $\dfrac{1}{a_{k+1}} = \dfrac{1}{a_k} - \dfrac{1}{a_k + n}$ and so

$$\sum_{k=0}^{n-1}\left(\frac{1}{a_k} - \frac{1}{a_{k+1}}\right) = \sum_{k=0}^{n-1}\frac{1}{n + a_k} < 1 \Rightarrow 2 - \frac{1}{a_n} < 1 \Rightarrow a_n < 1.$$

Now, because the sequence is increasing we also have $2 - \dfrac{1}{a_n} = \displaystyle\sum_{k=0}^{n-1}\frac{1}{n + a_k} > \frac{n}{n+1}$ and from this inequality and the previous one we conclude immediately that $1 - \dfrac{1}{n} < a_n < 1$.

46. [Călin Popa] Let a, b, c be positive real numbers, with $a, b, c \in (0,1)$ such that $ab + bc + ca = 1$. Prove that

$$\frac{a}{1-a^2} + \frac{b}{1-b^2} + \frac{c}{1-c^2} \geq \frac{3}{4}\left(\frac{1-a^2}{a} + \frac{1-b^2}{b} + \frac{1-c^2}{c}\right).$$

Solution:

It is known that in every triangle ABC the following identity holds $\sum \tan\frac{A}{2}\tan\frac{B}{2} = 1$ and because tan is bijective on $\left[0, \frac{\pi}{2}\right)$ we can set $a = \tan\frac{A}{2}$, $b = \tan\frac{B}{2}$, $c = \tan\frac{C}{2}$. The condition $a, b, c \in (0,1)$ tells us that the triangle ABC is acute-angled. With this substitution, the inequality becomes

$$\sum \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}} \geq 3\sum \frac{1-\tan^2\frac{A}{2}}{2\tan\frac{A}{2}} \Leftrightarrow \sum \tan A \geq 3\sum \frac{1}{\tan A} \Leftrightarrow$$

$$\Leftrightarrow \tan A\tan B\tan C(\tan A+\tan B+\tan C) \geq 3(\tan A\tan B+\tan B\tan C+\tan C\tan A) \Leftrightarrow$$

$$\Leftrightarrow (\tan A+\tan B+\tan C)^2 \geq 3(\tan A\tan B+\tan B\tan C+\tan C\tan A),$$

clearly true because $\tan A, \tan B, \tan C > 0$.

47. [Titu Andreescu, Gabriel Dospinescu] Let $x, y, z \leq 1$ and $x + y + z = 1$. Prove that

$$\frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+z^2} \leq \frac{27}{10}.$$

Solution:

Using the fact that $(4-3t)(1-3t)^2 \geq 0$ for any $t \leq 1$, we find that

$$\frac{1}{1+x^2} \leq \frac{27}{50}(2-x).$$

Writing two similar expressions for y and z and adding them up, we find the desired inequality.

Remark.

Tough it may seem too easy, this problem helps us to prove the following difficult inequality

$$\sum \frac{(b+c-a)^2}{a^2+(b+c)^2} \geq \frac{3}{5}.$$

In fact, this problem is equivalent to that difficult one. Try to prove this!

48. [Gabriel Dospinescu] Prove that if $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$, then

$$(1-x)^2(1-y)^2(1-z)^2 \geq 2^{15}xyz(x+y)(y+z)(z+x)$$

.

Solution:

We put $a = \sqrt{x}$, $b = \sqrt{y}$ and $c = \sqrt{z}$. Then $1 - x = 1 - a^2 = (a+b+c)^2 - a^2 = (b+c)(2a+b+c)$. Now we have to prove that $((a+b)(b+c)(c+a)(2a+b+c)(a+2b+c)(a+b+2c))^2 \geq 2^{15}a^2b^2c^2(a^2+b^2)(b^2+c^2)(c^2+a^2)$. But this inequality is true as it follows from the following true inequalities $ab(a^2+b^2) \le \dfrac{(a+b)^4}{8}$ (this being equivalent to $(a-b)^4 \ge 0$) and $(2a+b+c)(a+2b+c)(a+b+2c) \ge 8(b+c)(c+a)(a+b)$.

49. Let x, y, z be positive real numbers such that $xyz = x+y+z+2$. Prove that

(1) $xy+yz+zx \ge 2(x+y+z)$;

(2) $\sqrt{x}+\sqrt{y}+\sqrt{z} \le \dfrac{3}{2}\sqrt{xyz}$.

***Solution*:**

The initial condition $xyz = x+y+z+2$ can be rewritten as follows

$$\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}=1.$$

Now, let

$$\frac{1}{1+x}=a,\ \frac{1}{1+y}=b,\ \frac{1}{1+z}=c.$$

Then

$$x=\frac{1-a}{a}=\frac{b+c}{a},\ y=\frac{c+a}{b},\ z=\frac{a+b}{c}.$$

(1) We have

$$\begin{aligned} xy+yz+zx &\ge 2(x+y+z) \Leftrightarrow \frac{b+c}{a}\cdot\frac{c+a}{b}+\frac{c+a}{b}\cdot\frac{a+b}{c}+\frac{a+b}{c}\cdot\frac{b+c}{a} \ge \\ &\ge 2\left(\frac{b+c}{a}+\cdot\frac{c+a}{b}+\frac{a+b}{c}\right) \Leftrightarrow a^3+b^3+c^3+3abc \ge \\ &\ge ab(a+b)+bc(b+c)+ca(c+a) \Leftrightarrow \sum a(a-b)(a-c) \ge 0, \end{aligned}$$

which is exactly **Schur's Inequality**.

(2) Here we have

$$\sqrt{x}+\sqrt{y}+\sqrt{z} \le \frac{3}{2}\sqrt{xyz} \Leftrightarrow$$

$$\Leftrightarrow \sqrt{\frac{a}{b+c}\cdot\frac{b}{c+a}}+\sqrt{\frac{b}{c+a}\cdot\frac{c}{a+b}}+\sqrt{\frac{c}{a+b}\cdot\frac{a}{b+c}} \le \frac{3}{2}.$$

This can be proved by adding the inequality

$$\sqrt{\frac{a}{b+c}\cdot\frac{b}{c+a}} \le \frac{1}{2}\left(\frac{a}{a+c}+\frac{b}{b+c}\right),$$

with the analogous ones.

50. Prove that if x, y, z are real numbers such that $x^2+y^2+z^2=2$, then

$$x+y+z \le xyz+2.$$

IMO Shortlist, 1987

First solution:

If one of x, y, z is negative, let us say x then

$$2 + xyz - x - y - z = (2 - y - z) - x(1 - yz) \geq 0,$$

because $y + z \leq \sqrt{2(y^2 + z^2)} \leq 2$ and $zy \leq \dfrac{z^2 + y^2}{2} \leq 1$. So we may assume that $0 < x \leq y \leq z$. If $z \leq 1$ then

$$2 + xyz - x - y - z = (1 - z)(1 - xy) + (1 - x)(1 - y) \geq 0.$$

Now, if $z > 1$ we have

$$z + (x + y) \leq \sqrt{2(z^2 + (x + y)^2)} = 2\sqrt{1 + xy} \leq 2 + xy \leq 2 + xyz.$$

This ends the proof.

Second solution:

Using the **Cauchy-Schwarz Inequality**, we find that

$$x + y + z - xyz = x(1 - yz) + y + z \leq \sqrt{(x^2 + (y + z)^2) \cdot (1 + (1 - yz)^2)}.$$

So, it is enough to prove that this last quantity is at most 2, which is equivalent to the inequality $(2 + 2yz)(2 - 2yz + (yz)^2) \leq 4 \Leftrightarrow 2(yz)^3 \leq 2(yz)^2$, which is clearly true, because $2 \geq y^2 + z^2 \geq 2yz$.

51. [Titu Andreescu, Gabriel Dospinescu] Prove that for any $x_1, x_2, \ldots, x_n \in (0, 1)$ and for any permutation σ of the set $\{1, 2, \ldots, n\}$, we have the inequality

$$\sum_{i=1}^{n} \frac{1}{1 - x_i} \geq \left(1 + \frac{\sum\limits_{i=1}^{n} x_i}{n} \right) \cdot \left(\sum_{i=1}^{n} \frac{1}{1 - x_i \cdot x_{\sigma(i)}} \right).$$

Solution:

Using the **AM-GM Inequality** and the fact that $\dfrac{1}{x + y} \leq \dfrac{1}{4x} + \dfrac{1}{4y}$, we can write the following chain of inequalities

$$\begin{aligned}\sum_{i=1}^{n} \frac{1}{1 - x_i \cdot y_i} &\leq \sum_{i=1}^{n} \frac{1}{1 - \dfrac{x_i^2 + y_i^2}{2}} = 2\sum_{i=1}^{n} \frac{1}{1 - x_i^2 + 1 - y_i^2} \leq \\ &\leq \sum_{i=1}^{n} \left(\frac{1}{2(1 - x_i^2)} + \frac{1}{2(1 - y_i^2)} \right) = \sum_{i=1}^{n} \frac{1}{1 - x_i^2}.\end{aligned}$$

So, it is enough to prove the inequality

$$\sum_{i=1}^{n} \frac{1}{1-x_i} \geq \left(1+\frac{\sum_{i=1}^{n} x_i}{n}\right) \cdot \left(\sum_{i=1}^{n} \frac{1}{1-x_i^2}\right) \Leftrightarrow$$

$$\Leftrightarrow \sum_{i=1}^{n} \frac{x_i}{1-x_i^2} \geq \frac{1}{n} \cdot \left(\sum_{i=1}^{n} x_i\right) \cdot \left(\sum_{i=1}^{n} \frac{1}{1-x_i^2}\right),$$

which is **Chebyshev Inequality** for the systems $(x_1, x_2, \ldots, x_n)$ and $\left(\frac{1}{1-x_1^2}, \frac{1}{1-x_2^2}, \ldots, \frac{1}{1-x_n^2}\right)$.

52. Let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $\sum_{i=1}^{n} \frac{1}{1+x_i} = 1$. Prove that

$$\sum_{i=1}^{n} \sqrt{x_i} \geq (n-1) \sum_{i=1}^{n} \frac{1}{\sqrt{x_i}}.$$

Vojtech Jarnik

***First solution*:**

Let $\frac{1}{1+x_i} = a_i$. The inequality becomes

$$\sum_{i=1}^{n} \sqrt{\frac{1-a_i}{a_i}} \geq (n-1) \sum_{i=1}^{n} \sqrt{\frac{a_i}{1-a_i}} \Leftrightarrow \sum_{i=1}^{n} \frac{1}{\sqrt{a_i(1-a_i)}} \geq$$

$$\geq n \sum_{i=1}^{n} \sqrt{\frac{a_i}{1-a_i}} \Leftrightarrow n \sum_{i=1}^{n} \sqrt{\frac{a_i}{1-a_i}} \leq \left(\sum_{i=1}^{n} a_i\right) \left(\sum_{i=1}^{n} \frac{1}{\sqrt{a_i(1-a_i)}}\right).$$

But the last inequality is a consequence of **Chebyshev's Inequality** for the n-tuples $(a_1, a_2, \ldots, a_n)$ and

$$\left(\frac{1}{\sqrt{a_1(1-a_1)}}, \frac{1}{\sqrt{a_2(1-a_2)}}, \ldots, \frac{1}{\sqrt{a_n(1-a_n)}}\right).$$

***Solution 2*:**

With the same notations, we have to prove that

$$(n-1) \sum_{i=1}^{n} \sqrt{\frac{a_i}{a_1+a_2+\ldots+a_{i-1}+a_{i+1}+\ldots+a_n}} \leq$$

$$\leq \sum_{i=1}^{n} \sqrt{\frac{a_1+a_2+\ldots+a_{i-1}+a_{i+1}+\ldots+a_n}{a_i}}.$$

But using the **Cauchy-Schwarz Inequality** and the **AM-GM Inequality**, we deduce that

$$\sum_{i=1}^{n}\sqrt{\frac{a_1+a_2+\ldots+a_{i-1}+a_{i+1}+\ldots+a_n}{a_i}}\geq$$

$$\begin{aligned}
&\geq \sum_{i=1}^{n}\frac{\sqrt{a_1}+\sqrt{a_2}+\ldots+\sqrt{a_{i-1}}+\sqrt{a_{i+1}}+\ldots+\sqrt{a_n}}{\sqrt{n-1}\cdot\sqrt{a_i}}=\\
&= \sum_{i=1}^{n}\frac{\sqrt{a_i}}{\sqrt{n-1}}\left(\frac{1}{\sqrt{a_1}}+\frac{1}{\sqrt{a_2}}+\cdots+\frac{1}{\sqrt{a_{i-1}}}+\frac{1}{\sqrt{a_{i+1}}}+\cdots+\frac{1}{\sqrt{a_n}}\right)\geq\\
&\geq \sum_{i=1}^{n}\frac{(n-1)\sqrt{n-1}\cdot\sqrt{a_i}}{\sqrt{a_1}+\sqrt{a_2}+\cdots+\sqrt{a_{i-1}}+\sqrt{a_{i+1}}+\cdots+\sqrt{a_n}}\geq\\
&\geq \sum_{i=1}^{n}(n-1)\sqrt{\frac{a_i}{a_1+a_2+\cdots+a_{i-1}+a_{i+1}+\cdots+a_n}}
\end{aligned}$$

and we are done.

53. [Titu Andreescu] Let $n>3$ and $a_1,a_2,\ldots,a_n$ be real numbers such that $a_1+a_2+\ldots+a_n\geq n$ and $a_1^2+a_2^2+\ldots+a_n^2\geq n^2$. Prove that $max\{a_1,a_2,\ldots,a_n\}\geq 2$.

USAMO, 1999

Solution:

The most natural idea is to suppose that $a_i<2$ for all i and to substitute $x_i=2-a_i>0$. Then we have $\sum_{i=1}^{n}(2-x_i)\geq n\Rightarrow\sum_{i=1}^{n}x_i\leq n$ and also

$$n^2\leq\sum_{i=1}^{n}a_i^2=\sum_{i=1}^{n}(2-x_i)^2=4n-4\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}x_i^2$$

Now, using the fact that $x_i>0$, we obtain $\sum_{i=1}^{n}x_i^2<\left(\sum_{i=1}^{n}x_i\right)^2$, which combined with the above inequality yields

$$n^2<4n-4\sum_{i=1}^{n}x_i+\left(\sum_{i=1}^{n}x_i\right)^2<4n+(n-4)\sum_{i=1}^{n}x_i$$

(we have used the fact that $\sum_{i=1}^{n}x_i\leq n$). Thus, we have $(n-4)\left(\sum_{i=1}^{n}x_i-n\right)>0$, which is clearly impossible since $n\geq 4$ and $\sum_{i=1}^{n}x_i\leq n$. So, our assumption was wrong and consequently $max\{a_1,a_2,\ldots,a_n\}\geq 2$.

54. [Vasile Cîrtoaje] If a, b, c, d are positive real numbers, then

$$\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+a}+\frac{d-a}{a+b}\geq 0.$$

***Solution*:**

We have

$$\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+a}+\frac{d-a}{a+b}=\frac{a+c}{b+c}+\frac{b+d}{c+d}+\frac{c+a}{d+a}+\frac{d+b}{a+b}-4=$$
$$=(a+c)\left(\frac{1}{b+c}+\frac{1}{d+a}\right)+(b+d)\left(\frac{1}{c+d}+\frac{1}{a+b}\right)-4.$$

Since

$$\frac{1}{b+c}+\frac{1}{d+a}\geq\frac{4}{(b+c)+(d+a)},\ \frac{1}{c+d}+\frac{1}{a+b}\geq\frac{4}{(c+d)+(a+b)},$$

we get

$$\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+a}+\frac{d-a}{a+b}\geq\frac{4(a+c)}{(b+c)+(d+a)}+\frac{4(b+d)}{(c+d)+(a+b)}-4=0.$$

Equality holds for $a=c$ and $b=d$.

Conjecture (Vasile Cîrtoaje)

If a, b, c, d, e are positive real numbers, then

$$\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+e}+\frac{d-e}{e+a}+\frac{e-a}{a+b}\geq 0.$$

55. If x and y are positive real numbers, show that $x^y+y^x>1$.

France, 1996

***Solution*:**

We will prove that $a^b\geq\frac{a}{a+b-ab}$ for any $a, b\in(0,1)$. Indeed, from **Bernoulli Inequality** it follows that $a^{1-b}=(1+a-1)^{1-b}\leq 1+(a-1)(1-b)=a+b-ab$ and thus the conclusion. Now, it x or y is at least 1, we are done. Otherwise, let $0<x, y<1$. In this case we apply the above observation and find that $x^y+y^x\geq\frac{x}{x+y-xy}+\frac{y}{x+y-xy}>\frac{x}{x+y}+\frac{y}{x+y}=1$.

56. Prove that if $a, b, c>0$ have product 1, then

$$(a+b)(b+c)(c+a)\geq 4(a+b+c-1).$$

MOSP, 2001

First solution:

Using the identity $(a+b)(b+c)(c+a) = (a+b+c)(ab+bc+ca) - 1$ we reduce the problem to the following one

$$ab + bc + ca + \frac{3}{a+b+c} \geq 4.$$

Now, we can apply the **AM-GM Inequality** in the following form

$$ab + bc + ca + \frac{3}{a+b+c} \geq 4\sqrt[4]{\frac{(ab+bc+ca)^3}{9(a+b+c)}}.$$

And so it is enough to prove that

$$(ab+bc+ca)^3 \geq 9(a+b+c).$$

But this is easy, because we clearly have $ab + bc + ca \geq 3$ and $(ab+bc+ca)^2 \geq 3abc(a+b+c) = 3(a+b+c)$.

Second solution:

We will use the fact that $(a+b)(b+c)(c+a) \geq \frac{8}{9}(a+b+c)(ab+bc+ca)$. So, it is enough to prove that $\frac{2}{9}(ab+bc+ca) + \frac{1}{a+b+c} \geq 1$. Using the **AM-GM Inequality**, we can write

$$\frac{2}{9}(ab+bc+ca) + \frac{1}{a+b+c} \geq 3\sqrt[3]{\frac{(ab+bc+ca)^2}{81(a+b+c)}} \geq 1,$$

because

$$(ab+bc+ca)^2 \geq 3abc(a+b+c) = 3(a+b+c).$$

57. Prove that for any $a, b, c > 0$,

$$(a^2+b^2+c^2)(a+b-c)(b+c-a)(c+a-b) \leq abc(ab+bc+ca).$$

Solution:

Clearly, if one of the factors in the left-hand side is negative, we are done. So, we may assume that a, b, c are the side lenghts of a triangle ABC. With the usual notations in a triangle, the inequality becomes

$$(a^2+b^2+c^2)\cdot\frac{16K^2}{a+b+c} \leq abc(ab+bc+ca) \Leftrightarrow (a+b+c)(ab+bc+ca)R^2 \geq abc(a^2+b^2+c^2).$$

But this follows from the fact that $(a+b+c)(ab+bc+ca) \geq 9abc$ and

$$0 \leq OH^2 = 9R^2 - a^2 - b^2 - c^2.$$

58. [D.P.Mavlo] Let $a, b, c > 0$. Prove that

$$3 + a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3\frac{(a+1)(b+1)(c+1)}{1+abc}.$$

Kvant, 1988

***Solution*:**
The inequality is equivalent to the following one

$$\sum a + \sum \frac{1}{a} + \sum \frac{a}{b} \geq 3\frac{\sum ab + \sum a}{abc+1}$$

or

$$abc\sum a + \sum \frac{1}{a} + \sum a^2c + \sum \frac{a}{b} \geq 2\left(\sum a + \sum ab\right).$$

But this follows from the inequalities

$$a^2bc + \frac{b}{c} \geq 2ab, b^2ca + \frac{c}{a} \geq 2bc, c^2ab + \frac{a}{b} \geq 2ca$$

and

$$a^2c + \frac{1}{c} \geq 2a, b^2a + \frac{1}{a} \geq 2b, c^2b + \frac{1}{b} \geq 2c.$$

59. [Gabriel Dospinescu] Prove that for any positive real numbers $x_1, x_2, \ldots, x_n$ with product 1 we have the inequality

$$n^n \cdot \prod_{i=1}^{n} (x_i^n + 1) \geq \left(\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{1}{x_i}\right)^n.$$

***Solution*:**
Using the **AM-GM Inequality**, we deduce that

$$\frac{x_1^n}{1+x_1^n} + \frac{x_2^n}{1+x_2^n} + \cdots + \frac{x_{n-1}^n}{1+x_{n-1}^n} + \frac{1}{1+x_n^n} \geq \frac{n}{x_n \cdot \sqrt[n]{\prod_{i=1}^{n}(1+x_i^n)}}$$

and

$$\frac{1}{1+x_1^n} + \frac{1}{1+x_2^n} + \cdots + \frac{1}{1+x_{n-1}^n} + \frac{x_n^n}{1+x_n^n} \geq \frac{n \cdot x_n}{\sqrt[n]{\prod_{i=1}^{n}(1+x_i^n)}}$$

Thus, we have

$$\sqrt[n]{\prod_{i=1}^{n}(1+x_i^n)} \geq x_n + \frac{1}{x_n}.$$

Of course, this is true for any other variable, so we can add all these inequalities to obtain that

$$n\sqrt[n]{\prod_{i=1}^{n}(1+x_i^n)} \geq \sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\frac{1}{x_i}$$

which is the desired inequality.

60. Let $a, b, c, d > 0$ such that $a+b+c=1$. Prove that

$$a^3+b^3+c^3+abcd \geq \min\left\{\frac{1}{4}, \frac{1}{9}+\frac{d}{27}\right\}.$$

Kvant, 1993

***Solution*:**

Suppose the inequality is false. Then we have, taking into account that $abc \leq \frac{1}{27}$, the inequality $d\left(\frac{1}{27}-abc\right) > a^3+b^3+c^3-\frac{1}{9}$. We may assume that $abc < \frac{1}{27}$. Now, we will reach a contradiction proving that $a^3+b^3+c^3+abcd \geq \frac{1}{4}$. It is sufficient to prove that

$$\frac{a^3+b^3+c^3-\frac{1}{9}}{\frac{1}{27}-abc}abc + a^3+b^3+c^3 \geq \frac{1}{4}.$$

But this inequality is equivalent to $4\sum a^3 + 15abc \geq 1$. We use now the identity $\sum a^3 = 3abc+1-3\sum ab$ and reduce the problem to proving that $\sum ab \leq \frac{1+9abc}{4}$, which is **Schur's Inequality**.

61. Prove that for any real numbers a, b, c we have the inequality

$$\sum(1+a^2)^2(1+b^2)^2(a-c)^2(b-c)^2 \geq (1+a^2)(1+b^2)(1+c^2)(a-b)^2(b-c)^2(c-a)^2.$$

AMM

***Solution*:**

The inequality can be also written as $\sum \frac{(1+a^2)(1+b^2)}{(1+c^2)(a-b)^2} \geq 1$ (of course, we may assume that a, b, c are distinct). Now, adding the inequalities

$$\frac{(1+a^2)(1+b^2)}{(1+c^2)(a-b)^2} + \frac{(1+b^2)(1+c^2)}{(1+a^2)(b-c)^2} \geq 2\frac{1+b^2}{|a-b||c-b|}$$

(which can be found using the **AM-GM Inequality**) we deduce that

$$\sum \frac{(1+a^2)(1+b^2)}{(1+c^2)(a-b)^2} \geq \sum \frac{1+b^2}{|(b-a)(b-c)|}$$

and so it is enough to prove that the last quantity is at least 1. But it follows from

$$\sum \frac{1+b^2}{|(b-a)(b-c)|} \geq \left|\sum \frac{1+b^2}{(b-a)(b-c)}\right| = 1$$

and the problem is solved.

62. [Titu Andreescu, Mircea Lascu] Let α, x, y, z be positive real numbers such that $xyz = 1$ and $\alpha \geq 1$. Prove that

$$\frac{x^\alpha}{y+z} + \frac{y^\alpha}{z+x} + \frac{z^\alpha}{x+y} \geq \frac{3}{2}.$$

First solution:

We may of course assume that $x \geq y \geq z$. Then we have

$$\frac{x}{y+z} \geq \frac{y}{z+x} \geq \frac{z}{x+y}$$

and $x^{\alpha-1} \geq y^{\alpha-1} \geq z^{\alpha-1}$. Using **Chebyshev's Inequality** we infer that

$$\sum \frac{x^\alpha}{y+z} \geq \frac{1}{3} \cdot \left(\sum x^{\alpha-1}\right) \cdot \left(\sum \frac{x}{y+z}\right).$$

Now, all we have to do is to observe that this follows from the inequalities $\sum x^{\alpha-1} \geq 3$ (from the **AM-GM Inequality**) and $\sum \frac{x}{y+z} \geq \frac{3}{2}$

Second solution:

According to the **Cauchy-Schwarz Inequality**, we have:

$$[(x(y+z)+y(z+x)+z(x+y)]] \left(\frac{x^\alpha}{y+z} + \frac{y^\alpha}{z+x} + \frac{z^\alpha}{x+y}\right) \geq \left(x^{\frac{1+\alpha}{2}} + y^{\frac{1+\alpha}{2}} + z^{\frac{1+\alpha}{2}}\right)^2.$$

Thus it remains to show that

$$\left(x^{\frac{1+\alpha}{2}} + y^{\frac{1+\alpha}{2}} + z^{\frac{1+\alpha}{2}}\right)^2 \geq 3(xy+yz+zx).$$

Since $(x+y+z)^2 \geq 3(xy+yz+zx)$, it is enough to prove that

$$x^{\frac{1+\alpha}{2}} + y^{\frac{1+\alpha}{2}} + z^{\frac{1+\alpha}{2}} \geq x+y+z.$$

From **Bernoulli's Inequality**, we get

$$x^{\frac{1+\alpha}{2}} = [1+(x-1)]^{\frac{1+\alpha}{2}} \geq 1 + \frac{1+\alpha}{2}(x-1) = \frac{1-\alpha}{2} + \frac{1+\alpha}{2}x$$

and, similarly,

$$y^{\frac{1+\alpha}{2}} \geq \frac{1-\alpha}{2} + \frac{1+\alpha}{2}y, z^{\frac{1+\alpha}{2}} \geq \frac{1-\alpha}{2} + \frac{1+\alpha}{2}z.$$

Thus

$$x^{\frac{1+\alpha}{2}} + y^{\frac{1+\alpha}{2}} + z^{\frac{1+\alpha}{2}} - (x+y+z) \geq \frac{3(1-\alpha)}{2} + \frac{1+\alpha}{2}(x+y+z) - (x+y+z) =$$

$$\frac{\alpha-1}{2}(x+y+z-3) \geq \frac{\alpha-1}{2}(3\sqrt[3]{xyz}-3) = 0.$$

Equality holds for $x = y = z = 1$.

***Remark*:**

Using the substitution $\beta = \alpha + 1$ ($\beta \geq 2$) and $x = \frac{1}{a}$, $y = \frac{1}{b}$, $z = \frac{1}{c}$ ($abc = 1$) the inequality becomes as follows

$$\frac{1}{a^\beta(b+c)} + \frac{1}{b^\beta(c+a)} + \frac{1}{c^\gamma(a+b)} \geq \frac{3}{2}.$$

For $\beta = 3$, we obtain one of the problems of IMO 1995 (proposed by Russia).

63. Prove that for any real numbers $x_1, \ldots, x_n, y_1, \ldots, y_n$ such that $x_1^2 + \cdots + x_n^2 = y_1^2 + \cdots + y_n^2 = 1$,

$$(x_1y_2 - x_2y_1)^2 \leq 2\left(1 - \sum_{k=1}^{n} x_ky_k\right).$$

Korea, 2001

***Solution*:**

We clearly have the inequality

$$(x_1y_2 - x_2y_1)^2 \leq \sum_{1\leq i<j\leq n} (x_iy_j - x_jy_i)^2 = \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) - \left(\sum_{i=1}^{n} x_iy_i\right)^2 =$$

$$= \left(1 - \sum_{i=1}^{n} x_iy_i\right)\left(1 + \sum_{i=1}^{n} x_iy_i\right).$$

Because we also have $\left|\sum_{i=1}^{n} x_iy_i\right| \leq 1$, we find immediately that

$$\left(1 - \sum_{i=1}^{n} x_iy_i\right)\left(1 + \sum_{i=1}^{n} x_iy_i\right) \leq 2\left(1 - \sum_{i=1}^{n} x_iy_i\right)$$

and the problem is solved.

64. [Laurenţiu Panaitopol] Let $a_1, a_2, \ldots, a_n$ be pairwise distinct positive integers. Prove that

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq \frac{2n+1}{3}(a_1 + a_2 + \cdots + a_n).$$

TST Romania

Solution:

Without loss of generality, we may assume that $a_1 < a_2 < \cdots < a_n$ and hence $a_i \geq i$ for all i. Thus, we may take $b_i = a_i - i \geq 0$ and the inequality becomes

$$\sum_{i=1}^{n} b_i^2 + 2\sum_{i=1}^{n} ib_i + \frac{n(n+1)(2n+1)}{6} \geq \frac{2n+1}{3} \cdot \sum_{i=1}^{n} b_i + \frac{n(n+1)(2n+1)}{6}.$$

Now, using the fact that $a_{i+1} > a_i$ we infer that $b_1 \leq b_2 \leq \cdots \leq b_n$ and from **Chebyshev's Inequality** we deduce that

$$2\sum_{i=1}^{n} ib_i \geq (n+1)\sum_{i=1}^{n} b_i \geq \frac{2n+1}{3}\sum_{i=1}^{n} b_i$$

and the conclusion is immediate. Also, from the above relations we can see immediately that we have equality if and only if $a_1, a_2, \ldots, a_n$ is a permutation of the numbers $1, 2, \ldots, n$.

65. [Călin Popa] Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\frac{b\sqrt{c}}{a(\sqrt{3c}+\sqrt{ab})} + \frac{c\sqrt{a}}{b(\sqrt{3a}+\sqrt{bc})} + \frac{a\sqrt{b}}{c(\sqrt{3b}+\sqrt{ca})} \geq \frac{3\sqrt{3}}{4}.$$

Solution:

Rewrite the inequality in the form

$$\sum \frac{\sqrt{\dfrac{bc}{a}}}{\sqrt{\dfrac{3ca}{b}} + a} \geq \frac{3\sqrt{3}}{4}.$$

With the substitution $x = \sqrt{\dfrac{bc}{a}}, y = \sqrt{\dfrac{ca}{b}}, z = \sqrt{\dfrac{ab}{c}}$ the condition $a + b + c = 1$ becomes $xy + yz + zx = 1$ and the inequality turns into

$$\sum \frac{x}{\sqrt{3}y + yz} \geq \frac{3\sqrt{3}}{4}.$$

But, by applying the **Cauchy-Schwarz Inequality** we obtain

$$\sum \frac{x^2}{\sqrt{3}xy + xyz} \geq \frac{\left(\sum x\right)^2}{\sqrt{3} + 3xyz} \geq \frac{3\sum xy}{\sqrt{3} + \dfrac{1}{\sqrt{3}}} = \frac{3\sqrt{3}}{4},$$

where we used the inequalities

$$\left(\sum x\right)^2 \geq 3\left(\sum xy\right) \text{ and } xyz \leq \frac{1}{3\sqrt{3}}.$$

66. [Titu Andreescu, Gabriel Dospinescu] Let a, b, c, d be real numbers such that $(1+a^2)(1+b^2)(1+c^2)(1+d^2) = 16$. Prove that

$$-3 \le ab + bc + cd + da + ac + bd - abcd \le 5.$$

Solution:

Let us write the condition in the form $16 = \prod(i+a) \cdot \prod(a-i)$. Using symmetric sums, we can write this as follows

$$16 = \left(1 - i\sum a - \sum ab + i\sum abc + abcd\right)\left(1 + i\sum a - \sum ab - i\sum abc + abcd\right).$$

So, we have the identity $16 = (1 - \sum ab + abcd)^2 + (\sum a - \sum abc)^2$. This means that $|1 - \sum ab + abcd| \le 4$ and from here the conclusion follows.

67. Prove that

$$(a^2+2)(b^2+2)(c^2+2) \ge 9(ab+bc+ca)$$

for any positive real numbers a, b, c.

APMO, 2004

First solution:

We will prove even more: $(a^2+2)(b^2+2)(c^2+2) \ge 3(a+b+c)^2$. Because $(a+b+c)^2 \le (|a|+|b|+|c|)^2$, we may assume that a, b, c are nonnegative. We will use the fact that if x and y have the same sign then $(1+x)(1+y) \ge 1+x+y$. So, we write the inequality in the form

$$\prod\left(\frac{a^2-1}{3}+1\right) \ge \frac{(a+b+c)^2}{9}$$

and we have three cases

i) If a, b, c are at least 1, then $\prod\left(\frac{a^2-1}{3}+1\right) \ge 1 + \sum \frac{a^2-1}{3} \ge \frac{\left(\sum a\right)^2}{9}$.

ii) If two of the three numbers are at least 1, let them be a and b, then we have

$$\begin{aligned}\prod\left(\frac{a^2-1}{3}+1\right) &\ge \left(1 + \frac{a^2-1}{3} + \frac{b^2-1}{3}\right)\left(\frac{c^2+2}{3}\right) \\ &= \frac{(a^2+b^2+1)}{9}\frac{(1^2+1^2+c^2)}{9} \ge \frac{(a+b+c)^2}{9}\end{aligned}$$

by the **Cauchy-Schwarz Inequality**.

iii) If all three numbers are at most 1, then by **Bernoulli Inequality** we have

$$\prod\left(\frac{a^2-1}{3}+1\right) \ge 1 + \sum \frac{a^2-1}{3} \ge \frac{\left(\sum a\right)^2}{9}$$

and the proof is complete.

Second solution:

Expanding everything, we reduce the problem to proving that

$$(abc)^2 + 2\sum a^2b^2 + 4\sum a^2 + 8 \geq 9\sum ab.$$

Because $3\sum a^2 \geq 3\sum ab$ and $2\sum a^2b^2 + 6 \geq 4\sum ab$, we are left with the inequality $(abc)^2 + \sum a^2 + 2 \geq 2\sum ab$. Of course, we can assume that a, b, c are non-negative and we can write $a = x^2, b = y^2, c = z^2$. In this case

$$2\sum ab - \sum a^2 = (x+y+z)(x+y-z)(y+z-x)(z+x-y).$$

It is clear that if x, y, z are not side lengths of a triangle, then the inequality is trivial. Otherwise, we can take $x = u+v, y = v+w, z = w+u$ and reduce the inequality to

$$((u+v)(v+w)(w+u))^4 + 2 \geq 16(u+v+w)uvw.$$

We have $((u+v)(v+w)(w+u))^4 + 1 + 1 \geq 3\sqrt[3]{(u+v)^4(v+w)^4(u+w)^4}$ and it remains to prove that the last quantity is at least $16(u+v+w)uvw$. This comes down to

$$(u+v)^4(v+w)^4(w+u)^4 \geq \frac{16^3}{3^3}(uvw)^3(u+v+w)^3.$$

But this follows from the known inequalities

$$(u+v)(v+w)(w+u) \geq \frac{8}{9}(u+v+w)(uv+vw+wu),$$

$$(uv+vw+wu)^4 \geq 3^4(uvw)^{\frac{8}{3}},\ u+v+w \geq 3\sqrt[3]{uvw}.$$

Third solution:

In the same manner as in the **Second solution**, we reduce the problem to proving that

$$(abc)^2 + 2 \geq 2\sum ab - \sum a^2.$$

Now, using **Schur's Inequality**, we infer that

$$2\sum ab - \sum a^2 \leq \frac{9abc}{a+b+c}$$

and as an immediate consequence of the **AM-GM Inequality** we have

$$\frac{9abc}{a+b+c} \leq 3\sqrt[3]{(abc)^2}.$$

This shows that as soon as we prove that

$$(abc)^2 + 2 \geq 3\sqrt[3]{(abc)^2},$$

the problem is solved. But the last assertion follows from the **AM-GM Inequality**.

68. [Vasile Cîrtoaje] Prove that if $0 < x \le y \le z$ and $x + y + z = xyz + 2$, then
a) $(1 - xy)(1 - yz)(1 - xz) \ge 0$;
b) $x^2 y \le 1, x^3 y^2 \le \frac{32}{27}$.

***Solution*:**
a) We have

$$(1 - xy)(1 - yz) = 1 - xy - yz + xy^2 z = 1 - xy - yz + y(x + y + z - 2) = (y - 1)^2 \ge 0$$

and similarly

$$(1 - yz)(1 - zx) = (1 - z)^2 \ge 0, \quad (1 - zx)(1 - xy) = (1 - x)^2 \ge 0.$$

So the expressions $1 - xy$, $1 - yz$ and $1 - zx$ have the same sign.

b) We rewrite the relation $x+y+z = xyz+2$ as $(1-x)(1-y)+(1-z)(1-xy) = 0$. If $x > 1$ then $z \ge y \ge x > 1$ and so $(1 - x)(1 - y) + (1 - z)(1 - xy) > 0$, impossible. So we have $x \le 1$. Next we distinguish two cases 1) $xy \le 1$; 2) $xy > 1$.

1) $xy \le 1$. We have $x^2 y \le x \le 1$ and $x^3 y^2 \le x \le 1 < \frac{32}{27}$.

2) $xy > 1$. From $y \ge \sqrt{xy}$ we get $y > 1$. Next we rewrite the relation $x + y + z = xyz + 2$ as $x + y - 2 = (xy - 1)z$. Because $z \ge y$ gives $x + y - 2 \ge (xy - 1)y$, $(y - 1)(2 - x - xy) \ge 0$ so $2 \ge x(1 + y)$. Using the **AM-GM Inequality**, we have $1 + y \ge 2\sqrt{y}$ and $1 + y = 1 + \frac{y}{2} + \frac{y}{2} \ge 3\sqrt[3]{1 \cdot \frac{y}{2} \cdot \frac{y}{2}}$. Thus we have $2 \ge 2x\sqrt{y}$ and $2 \ge 3x\sqrt[3]{\frac{y^2}{4}}$, which means that $x^2 y \le 1$ and $x^3 y^2 \le \frac{32}{27}$.

The equality $x^2 y = 1$ takes place when $x = y = 1$ and the equality $x^3 y^2 = \frac{32}{27}$ takes place when $x = \frac{2}{3}$, $y = z = 2$.

69. [Titu Andreescu] Let a, b, c be positive real numbers such that $a+b+c \ge abc$. Prove that at least two of the inequalities

$$\frac{2}{a} + \frac{3}{b} + \frac{6}{c} \ge 6, \frac{2}{b} + \frac{3}{c} + \frac{6}{a} \ge 6, \frac{2}{c} + \frac{3}{a} + \frac{6}{b} \ge 6,$$

are true.

TST 2001, USA

***Solution*:**
The most natural idea is to male the substitution $\frac{1}{a} = x$, $\frac{1}{b} = y$, $\frac{1}{c} = z$. Thus, we have $x, y, z > 0$ and $xy + yz + zx \ge 1$ and we have to prove that at least two of the inequalities $2x+3y+6z \ge 6$, $2y+3z+6x \ge 6$, $2z+3x+6y \ge 6$ are true. Suppose this is not the case. Then we may assume that $2x+3y+6z < 6$ and $2z+3x+6y < 6$. Adding, we find that $5x+9y+8z < 12$. But we have $x \ge \frac{1 - yz}{y + z}$. Thus, $12 > \frac{5 - 5yz}{y + z} + 9y + 8z$

which is the same as $12(y+z) > 5+9y^2+8z^2+12yz \Leftrightarrow (2z-1)^2+(3y+2z-2)^2 < 0$, which is clearly impossible. Thus, the conclusion follows.

70. [Gabriel Dospinescu, Marian Tetiva] Let $x, y, z > 0$ such that

$$x+y+z = xyz.$$

Prove that

$$(x-1)(y-1)(z-1) \leq 6\sqrt{3}-10.$$

First solution:

Because of $x < xyz \Rightarrow yz > 1$ (and the similar relations $xz > 1$, $xy > 1$) at most one of the three numbers can be less than 1. In any of these cases ($x \leq 1$, $y \geq 1$, $z \geq 1$ or the similar ones) the inequality to prove is clear. The only case we still have to analyse is that when $x \geq 1$, $y \geq 1$ and $z \geq 1$.

In this situation denote

$$x-1 = a,\ y-1 = b,\ z-1 = c.$$

Then a, b, c are nonnegative real numbers and, because

$$x = a+1,\ y = b+1,\ z = c+1,$$

they satisfy

$$a+1+b+1+c+1 = (a+1)(b+1)(c+1),$$

which means

$$abc+ab+ac+bc = 2.$$

Now let $x = \sqrt[3]{abc}$; we have

$$ab+ac+bc \geq 3\sqrt[3]{abacbc} = 3x^2,$$

that's why we get

$$x^3+3x^2 \leq 2 \Leftrightarrow (x+1)\left(x^2+2x-2\right) \leq 0 \Leftrightarrow$$
$$\Leftrightarrow (x+1)\left(x+1+\sqrt{3}\right)\left(x+1-\sqrt{3}\right) \leq 0.$$

For $x \geq 0$, this yields

$$\sqrt[3]{abc} = x \leq \sqrt{3}-1,$$

or, equivalently

$$abc \leq \left(\sqrt{3}-1\right)^3,$$

which is exactly

$$(x-1)(y-1)(z-1) \leq 6\sqrt{3}-10.$$

The proof is complete.

Second solution:

Like in the first solution (and due to the symmetry) we may suppose that $x \geq 1$, $y \geq 1$; we can even assume that $x > 1$, $y > 1$ (for $x = 1$ the inequality is plain). Then we get $xy > 1$ and from the given condition we have

$$z = \frac{x+y}{xy-1}.$$

The relation to prove is

$$(x-1)(y-1)(z-1) \leq 6\sqrt{3} - 10 \Leftrightarrow$$
$$\Leftrightarrow 2xyz - (xy + xz + yz) \leq 6\sqrt{3} - 9,$$

or, with this expression of z,

$$2xy\frac{x+y}{xy-1} - xy - (x+y)\frac{x+y}{xy-1} \leq 6\sqrt{3} - 9 \Leftrightarrow$$
$$\Leftrightarrow (xy - x - y)^2 + (6\sqrt{3} - 10)\, xy \leq 6\sqrt{3} - 9,$$

after some calculations.

Now, we put $x = a + 1$, $y = b + 1$ and transform this into

$$a^2b^2 + \left(6\sqrt{3} - 10\right)(a + b + ab) - 2ab \geq 0.$$

But

$$a + b \geq 2\sqrt{ab}$$

and $6\sqrt{3} - 10 > 0$, so it suffices to show that

$$a^2b^2 + \left(6\sqrt{3} - 10\right)\left(2\sqrt{ab} + ab\right) - 2ab \geq 0.$$

The substitution $t = \sqrt{ab} \geq 0$ reduces this inequality to

$$t^4 + \left(6\sqrt{3} - 12\right)t^2 + 2\left(6\sqrt{3} - 10\right)t \geq 0,$$

or

$$t^3 + \left(6\sqrt{3} - 12\right)t + 2\left(6\sqrt{3} - 10\right) \geq 0.$$

The derivative of the function

$$f(t) = t^3 + \left(6\sqrt{3} - 12\right)t + 2\left(6\sqrt{3} - 10\right),\ t \geq 0$$

is

$$f'(t) = 3\left(t^2 - \left(\sqrt{3} - 1\right)^2\right)$$

and has only one positive zero. It is $\sqrt{3} - 1$ and it's easy to see that this is a minimum point for f in the interval $[0, \infty)$. Consequently

$$f(t) \geq f\left(\sqrt{3} - 1\right) = 0,$$

and we are done.

A final observation: in fact we have

$$f(t) = \left(t - \sqrt{3} + 1\right)^2\left(t + 2\sqrt{3} - 2\right),$$

which shows that $f(t) \geq 0$ for $t \geq 0$.

71. [Marian Tetiva] Prove that for any positive real numbers a, b, c,

$$\left|\frac{a^3-b^3}{a+b}+\frac{b^3-c^3}{b+c}+\frac{c^3-a^3}{c+a}\right| \leq \frac{(a-b)^2+(b-c)^2+(c-a)^2}{4}.$$

Moldova TST, 2004

***First solution*:**

First of all, we observe that right hand side can be transformed into

$$\sum \frac{a^3-b^3}{a+b} = (a^3-b^3)\left(\frac{1}{a+b}-\frac{1}{a+c}\right)+(b^3-c^3)\left(\frac{1}{b+c}-\frac{1}{a+c}\right) =$$

$$= \frac{(a-b)(c-b)(a-c)\left(\sum ab\right)}{(a+b)(a+c)(b+c)}$$

and so we have to prove the inequality

$$\frac{|(a-b)(b-c)(c-a)|(ab+bc+ca)}{(a+b)(b+c)(c+a)} \leq \frac{1}{2}\left(\sum a^2 - \sum ab\right).$$

It is also easy to prove that $(a+b)(b+c)(c+a) \geq \frac{8}{9}(a+b+c)(ab+bc+ca)$ and so we are left with

$$\frac{2}{9}\cdot\sum a \cdot \left(\sum (a-b)^2\right) \geq \left|\prod (a-b)\right|.$$

Using the **AM-GM Inequality**, we reduce this inequality to the following one

$$\frac{8}{27}\left(\sum a\right)^3 \geq \left|\prod (a-b)\right|.$$

This one is easy. Just observe that we can assume that $a \geq b \geq c$ and in this case it becomes

$$(a-b)(a-c)(b-c) \leq \frac{8}{27}(a+b+c)^3$$

and it follows from the **AM-GM Inequality**.

***Second solution (by Marian Tetiva)*:**

It is easy to see that the inequality is not only cyclic, but symmetric. That is why we may assume that $a \geq b \geq c > 0$. The idea is to use the inequality

$$x+\frac{y}{2} \geq \frac{x^2+xy+y^2}{x+y} \geq y+\frac{x}{2},$$

which is true if $x \geq y > 0$. The proof of this inequality is easy and we won't insist. Now, because $a \geq b \geq c > 0$, we have the three inequalities

$$a+\frac{b}{2} \geq \frac{a^2+ab+b^2}{a+b} \geq b+\frac{a}{2}, b+\frac{c}{2} \geq \frac{b^2+bc+c^2}{b+c} \geq c+\frac{b}{2}$$

and of course

$$a+\frac{c}{2} \geq \frac{a^2+ac+c^2}{a+c} \geq c+\frac{a}{2}.$$

That is why we can write

$$\begin{aligned}\sum \frac{a^3-b^3}{a+b} &= (a-b)\frac{a^2+ab+b^2}{a+b}+(b-c)\frac{b^2+bc+c^2}{b+c}-(a-c)\frac{a^2+ac+c^2}{a+c} \geq \\ &\geq (a-b)\left(b+\frac{a}{2}\right)+(b-c)\left(c+\frac{b}{2}\right)-(a-c)\left(a+\frac{c}{2}\right)= \\ &= -\frac{\sum(a-b)^2}{4}.\end{aligned}$$

In the same manner we can prove that

$$\sum \frac{a^3-b^3}{a+b} \leq \frac{\sum(a-b)^2}{4}$$

and the conclusion follows.

72. [Titu Andreescu] Let a, b, c be positive real numbers. Prove that

$$(a^5-a^2+3)(b^5-b^2+3)(c^5-c^2+3) \geq (a+b+c)^3.$$

USAMO, 2004

Solution:

We start with the inequality $a^5-a^2+3 \geq a^3+2 \Leftrightarrow (a^2-1)(a^3-1) \geq 0$. Thus, it remains to show that

$$\prod (a^3+2) \geq \left(\sum a\right)^3.$$

Using the **AM-GM Inequality**, one has

$$\frac{a^3}{a^3+2}+\frac{1}{b^3+2}+\frac{1}{c^3+2} \geq \frac{3a}{\sqrt[3]{\prod(a^3+2)}}.$$

We write two similar inequalities and then add up all these relations. We will find that

$$\prod(a^3+2) \geq \left(\sum a\right)^3,$$

which is what we wanted.

73. [Gabriel Dospinescu] Let $n > 2$ and $x_1, x_2, \ldots, x_n > 0$ such that

$$\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} \frac{1}{x_k}\right) = n^2+1.$$

Prove that

$$\left(\sum_{k=1}^{n} x_k^2\right) \cdot \left(\sum_{k=1}^{n} \frac{1}{x_k^2}\right) > n^2+4+\frac{2}{n(n-1)}.$$

Solution:

In this problem, a combination between identities and the **Cauchy-Schwarz Inequality** is the way to proceed. So, let us start with the expression

$$\sum_{1\le i<j\le n}\left(\frac{x_i}{x_j}+\frac{x_j}{x_i}-2\right)^2.$$

We can immediately see that

$$\begin{aligned}\sum_{1\le i<j\le n}\left(\frac{x_i}{x_j}+\frac{x_j}{x_i}-2\right)^2 &= \sum_{1\le i<j\le n}\left(\frac{x_i^2}{x_j^2}+\frac{x_j^2}{x_i^2}-4\cdot\frac{x_i}{x_j}-4\cdot\frac{x_j}{x_i}+6\right)=\\ &= \left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n\frac{1}{x_i^2}\right)-4\left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n\frac{1}{x_i}\right)+3n^2.\end{aligned}$$

Thus we could find from the inequality

$$\sum_{1\le i<j\le n}\left(\frac{x_i}{x_j}+\frac{x_j}{x_i}-2\right)^2\ge 0$$

that

$$\left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n\frac{1}{x_i^2}\right)\ge n^2+4.$$

Unfortunately, this is not enough. So, let us try to minimize

$$\sum_{1\le i<j\le n}\left(\frac{x_i}{x_j}+\frac{x_j}{x_i}-2\right)^2.$$

This could be done using the **Cauchy-Schwarz Inequality**:

$$\sum_{1\le i<j\le n}\left(\frac{x_i}{x_j}+\frac{x_j}{x_i}-2\right)^2\ge\frac{\left(\sum\limits_{1\le i<j\le n}\left(\frac{x_i}{x_j}+\frac{x_j}{x_i}-2\right)\right)^2}{\binom{n}{2}}.$$

Because $\sum\limits_{i\le i<j\le n}\left(\frac{x_i}{x_j}+\frac{x_j}{x_i}-2\right)=1$, we deduce that

$$\left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n\frac{1}{x^2}\right)\ge n^2+4+\frac{2}{n(n-1)},$$

which is what we wanted to prove. Of course, we should prove that we cannot have equality. But equality would imply that $x_1=x_2=\cdots=x_n$, which contradicts the assumption

$$\left(\sum_{k=1}^n x_k\right)\left(\sum_{k=1}^n\frac{1}{x_k}\right)=n^2+1.$$

74. [Gabriel Dospinescu, Mircea Lascu, Marian Tetiva] Prove that for any positive real numbers a, b, c,

$$a^2 + b^2 + c^2 + 2abc + 3 \geq (1 + a)(1 + b)(1 + c).$$

First solution:

Let $f(a, b, c) = a^2 + b^2 + c^2 + abc + 2 - a - b - c - ab - bc - ca$. We have to prove that all values of f are nonnegative. If $a, b, c > 3$, then we clearly have $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$, which means that $f(a, b, c) > a^2 + b^2 + c^2 + 2 - a - b - c > 0$. So, we may assume that $a \leq 3$ and let $m = \frac{b + c}{2}$. Easy computations show that $f(a, b, c) - f(a, m, m) = \frac{(3 - a)(b - c)^2}{4} \geq 0$ and so it remains to prove that $f(a, m, m) \geq 0$, which is the same as

$$(a + 1)m^2 - 2(a + 1)m + a^2 - a + 2 \geq 0.$$

This is clearly true, because the discriminant of the quadratic equation is $-4(a + 1)(a - 1)^2 \leq 0$.

Second solution:

Recall **Turkevici's Inequality**

$$x^4 + y^4 + z^4 + t^4 + 2xyzt \geq x^2y^2 + y^2z^2 + z^2t^2 + t^2x^2 + x^2z^2 + y^2t^2$$

for all positive real numbers x, y, z, t. Taking $t = 1, a = x^2, b = y^2, x = z^2$ and using the fact that $2\sqrt{abc} \leq abc + 1$, we find the desired inequality.

75. [Titu Andreescu, Zuming Feng] Let a, b, c be positive real numbers. Prove that

$$\frac{(2a + b + c)^2}{2a^2 + (b + c)^2} + \frac{(2b + a + c)^2}{2b^2 + (a + c)^2} + \frac{(2c + a + b)^2}{2c^2 + (a + b)^2} \leq 8.$$

USAMO, 2003

First solution:

Because the inequality is homogeneous, we can assume that $a + b + c = 3$. Then

$$\frac{(2a + b + c)^2}{2a^2 + (b + c)^2} = \frac{a^2 + 6a + 9}{3a^2 - 6a + 9} = \frac{1}{3}\left(1 + 2 \cdot \frac{4a + 3}{2 + (a - 1)^2}\right) \leq$$

$$\leq \frac{1}{3}\left(1 + 2 \cdot \frac{4a + 3}{2}\right) = \frac{4a + 4}{3}.$$

Thus

$$\sum \frac{(2a + b + c)^2}{2a^2 + (b + c)^2} \leq \frac{1}{3}\sum(4a + 4) = 8.$$

Second solution:

Denote $x = \dfrac{b+c}{a}$, $y = \dfrac{c+a}{b}$, $z = \dfrac{a+b}{c}$. We have to prove that

$$\sum \frac{(x+2)^2}{x^2+2} \leq 8 \Leftrightarrow \sum \frac{2x+1}{x^2+2} \leq \frac{5}{2} \Leftrightarrow \sum \frac{(x-1)^2}{x^2+2} \geq \frac{1}{2}.$$

But, from the **Cauchy-Schwarz Inequality**, we have

$$\sum \frac{(x-1)^2}{x^2+2} \geq \frac{(x+y+z-3)^2}{x^2+y^2+z^2+6}.$$

It remains to prove that

$$2(x^2+y^2+z^2+2xy+2yz+2zx-6x-6y-6z+9) \geq x^2+y^2+z^2+6 \Leftrightarrow$$

$$\Leftrightarrow x^2+y^2+z^2+4(xy+yz+zx)-12(x+y+z)+12 \geq 0.$$

Now $xy+yz+zx \geq 3\sqrt[3]{x^2y^2z^2} \geq 12$ (because $xyz \geq 8$), so we still have to prove that $(x+y+z)^2+24-12(x+y+z)+12 \geq 0$, which is equivalent to $(x+y+z-6)^2 \geq 0$, clearly true.

76. Prove that for any positive real numbers x, y and any positive integers m, n,

$$(n-1)(m-1)(x^{m+n}+y^{m+n})+(m+n-1)(x^m y^n+x^n y^m) \geq mn(x^{m+n-1}y+y^{m+n-1}x).$$

Austrian-Polish Competition, 1995

Solution:

We transform the inequality as follows:

$$mn(x-y)(x^{m+n-1}-y^{m+n-1}) \geq (m+n-1)(x^m-y^m)(x^n-y^n) \Leftrightarrow$$

$$\Leftrightarrow \frac{x^{m+n-1}-y^{m+n-1}}{(m+n-1)(x-y)} \geq \frac{x^m-y^m}{m(x-y)} \cdot \frac{x^n-y^n}{n(x-y)}$$

(we have assumed that $x > y$). The last relation can also be written

$$(x-y)\int_y^x t^{m+n-2}dt \geq \int_y^x t^{m-1}dt \cdot \int_y^x t^{n-1}dt$$

and this follows from **Chebyshev's Inequality for integrals**.

77. Let a, b, c, d, e be positive real numbers such that $abcde = 1$. Prove that

$$\frac{a+abc}{1+ab+abcd}+\frac{b+bcd}{1+bc+bcde}+\frac{c+cde}{1+cd+cdea}+\frac{d+dea}{1+de+deab}+\frac{e+eab}{1+ea+eabc} \geq \frac{10}{3}.$$

Crux Mathematicorum

Solution:

We consider the standard substitution

$$a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{t}, d = \frac{t}{u}, e = \frac{u}{x}$$

with $x, y, z, t, u > 0$. It is clear that

$$\frac{a+abc}{1+ab+abcd} = \frac{\frac{1}{y}+\frac{1}{t}}{\frac{1}{x}+\frac{1}{z}+\frac{1}{u}}.$$

Writing the other relations as well, and denoting $\frac{1}{x} = a_1, \frac{1}{y} = a_2, \frac{1}{z} = a_3, \frac{1}{t} = a_4, \frac{1}{u} = a_5$, we have to prove that if $a_i > 0$, then

$$\sum \frac{a_2+a_4}{a_1+a_3+a_5} \geq \frac{1}{3}.$$

Using the **Cauchy-Schwarz Inequality**, we minor the left-hand side with

$$\frac{4S^2}{2S^2-(a_2+a_4)^2-(a_1+a_4)^2-(a_3+a_5)^2-(a_2+a_5)^2-(a_1+a_3)^2},$$

where $S = \sum_{i=1}^{5} a_i$. By applying the **Cauchy-Schwarz Inequality** again for the denominator of the fraction, we obtain the conclusion.

78. [Titu Andreescu] Prove that for any $a, b, c, \in \left(0, \frac{\pi}{2}\right)$ the following inequality holds

$$\frac{\sin a \cdot \sin(a-b) \cdot \sin(a-c)}{\sin(b+c)} + \frac{\sin b \cdot \sin(b-c) \cdot \sin(b-a)}{\sin(c+a)} + \frac{\sin c \cdot \sin(c-a) \cdot \sin(c-b)}{\sin(a+b)} \geq 0.$$

TST 2003, USA

Solution:

Let $x = \sin a$, $y = \sin b$, $z = \sin c$. Then we have $x, y, z > 0$. It is easy to see that the following relations are true:

$$\sin a \cdot \sin(a-b) \cdot \sin(a-c) \cdot \sin(a+b) \cdot \sin(a+c) = x(x^2-y^2)(x^2-z^2)$$

Using similar relations for the other terms, we have to prove that:

$$\sum x(x^2-y^2)(x^2-y^2) \geq 0.$$

With the substitution $x = \sqrt{u}$, $y = \sqrt{v}$, $z = \sqrt{w}$ the inequality becomes $\sum \sqrt{u}(u-v)(u-w) \geq 0$. But this follows from **Schur's Inequality**.

79. Prove that if a, b, c are positive real numbers then,

$$\sqrt{a^4+b^4+c^4} + \sqrt{a^2b^2+b^2c^2+c^2a^2} \geq \sqrt{a^3b+b^3c+c^3a} + \sqrt{ab^3+bc^3+ca^3}.$$

KMO Summer Program Test, 2001

Solution:

It is clear that it suffices to prove the following inequalities

$$\sum a^4 + \sum a^2b^2 \geq \sum a^3b + \sum ab^3$$

and

$$\left(\sum a^4\right)\left(\sum a^2b^2\right) \geq \left(\sum a^3b\right)\left(\sum ab^3\right).$$

The first one follows from **Schur's Inequality**

$$\sum a^4 + abc\sum a \geq \sum a^3b + \sum ab^3$$

and the fact that

$$\sum a^2b^2 \geq abc\sum a.$$

The second one is a simple consequence of the **Cauchy-Schwarz Inequality**:

$$(a^3b+b^3c+c^3a)^2 \leq (a^2b^2+b^2c^2+c^2a^2)(a^4+b^4+c^4)$$

$$(ab^3+bc^3+ca^3)^2 \leq (a^2b^2+b^2c^2+c^2a^2)(a^4+b^4+c^4).$$

80. [Gabriel Dospinescu, Mircea Lascu] For a given $n > 2$ find the smallest constant k_n with the property: if $a_1, \ldots, a_n > 0$ have product 1, then

$$\frac{a_1a_2}{(a_1^2+a_2)(a_2^2+a_1)} + \frac{a_2a_3}{(a_2^2+a_3)(a_3^2+a_2)} + \cdots + \frac{a_na_1}{(a_n^2+a_1)(a_1^2+a_n)} \leq k_n.$$

***Solution*:**

Let us take first $a_1 = a_2 = \cdots = a_{n-1} = x, a_n = \frac{1}{x^{n-1}}$. We infer that

$$k_n \geq \frac{2x^{2n-1}}{(x^{n+1}+1)(x^{2n-1}+1)} + \frac{n-2}{(1+x)^2} > \frac{n-2}{(1+x)^2}$$

for all $x > 0$. Clearly, this implies $k_n \geq n-2$. Let us prove that $n-2$ is a good constant and the problem will be solved.

First, we will prove that $(x^2+y)(y^2+x) \geq xy(1+x)(1+y)$. Indeed, this is the same as $(x+y)(x-y)^2 \geq 0$. So, it suffices to prove that

$$\frac{1}{(1+a_1)(1+a_2)} + \frac{1}{(1+a_2)(1+a_3)} + \cdots + \frac{1}{(1+a_n)(1+a_1)} \leq n-2.$$

Now, we take $a_1 = \frac{x_1}{x_2}, \ldots, a_n = \frac{x_n}{x_1}$ and the above inequality becomes

$$\sum_{k=1}^{n}\left(1 - \frac{x_{k+1}x_{k+2}}{(x_k+x_{k+1})(x_{k+1}+x_{k+2})}\right) \geq 2,$$

which can be also written in the following from

$$\sum_{k=1}^{n} \frac{x_k}{x_k+x_{k+1}} + \frac{x_{k+1}^2}{(x_k+x_{k+1})(x_{k+1}+x_{k+2})} \geq 2.$$

Clearly,

$$\sum_{k=1}^{n} \frac{x_k}{x_k+x_{k+1}} \geq \sum_{k=1}^{n} \frac{x_k}{x_1+x_2+\cdots+x_n} = 1.$$

So, we have to prove the inequality

$$\sum_{k=1}^{n} \frac{x_{k+1}^2}{(x_k + x_{k+1})(x_{k+1} + x_{k+2})} \geq 1.$$

Using the **Cauchy-Schwarz Inequality**, we infer that

$$\sum_{k=1}^{n} \frac{x_{k+1}^2}{(x_k + x_{k+1})(x_{k+1} + x_{k+2})} \geq \frac{\left(\sum_{k+1}^{n} x_k\right)^2}{\sum_{k=1}^{n}(x_k + x_{k+1})(x_{k+1} + x_{k+2})}$$

and so it suffices to prove that

$$\left(\sum_{k=1}^{n} x_k\right)^2 \geq \sum_{k=1}^{n} x_k^2 + 2\sum_{k=1}^{n} x_k x_{k+1} + \sum_{k=1}^{n} x_k x_{k+2}.$$

But this one is equivalent to

$$2\sum_{1\leq i<j\leq n} x_i x_j \geq 2\sum_{k=1}^{n} x_k x_{k+1} + \sum_{k=1}^{n} x_k x_{k+2}$$

and it is clear. Thus, $k_n = n - 2$.

81. [Vasile Cîrtoaje] For any real numbers a, b, c, x, y, z prove that the inequality holds

$$ax + by + cz + \sqrt{(a^2 + b^2 + c^2)(x^2 + y^2 + z^2)} \geq \frac{2}{3}(a + b + c)(x + y + z).$$

Kvant, 1989

Solution:

Let us denote $t = \sqrt{\dfrac{x^2 + y^2 + z^2}{a^2 + b^2 + c^2}}$. Using the substitutions $x = tp$, $y = tq$ and $z = tr$, which imply

$$a^2 + b^2 + c^2 = p^2 + q^2 + r^2.$$

The given inequality becomes

$$ap + bq + cr + a^2 + b^2 + c^2 \geq \frac{2}{3}(a + b + c)(p + q + r),$$

$$(a + p)^2 + (b + q)^2 + (c + r)^2 \geq \frac{4}{3}(a + b + c)(p + q + r).$$

Since

$$4(a + b + c)(p + q + r) \leq [(a + b + c) + (p + q + r)]^2,$$

it suffices to prove that

$$(a + p)^2 + (b + q)^2 + (c + r)^2 \geq \frac{1}{3}[(a + p) + (b + q) + (c + r)]^2.$$

This inequality is clearly true.

82. [Vasile Cîrtoaje] Prove that the sides a, b, c of a triangle satisfy the inequality

$$3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-1\right) \geq 2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

First solution:

We may assume that c is the smallest among a, b, c. Then let $x = b - \frac{a+c}{2}$. After some computations, the inequality becomes

$$(3a-2c)x^2+\left(x+c-\frac{a}{4}\right)(a-c)^2 \geq 0 \Leftrightarrow (3a-2c)(2b-a-c)^2+(4b+2c-3a)(a-c)^2 \geq 0$$

which follows immediately from $3a \geq 2c, 4b+2c-3a = 3(b+c-a)+b-c > 0$.

Second solution:

Make the classical substitution $a = y+z, b = z+x, c = x+y$ and clear denominators. The problem reduces to proving that

$$x^3+y^3+z^3+2(x^2y+y^2z+z^2x) \geq 3(xy^2+yz^2+zx^2).$$

We can of course assume that x is the smallest among x, y, z. Then we can write $y = x+m, z = x+n$ with nonnegatives m and n. A short computation shows that the inequality reduces to $2x(m^2-mn+n^2)+m^3+n^3+2m^2n-3n^2m \geq 0$. All we need to prove is that $m^3+n^3+2m^2n \geq 3n^2m \Leftrightarrow (n-m)^3-(n-m)m^2+m^3 \geq 0$ and this follows immediately from the inequality $t^3+1 \geq 3$, true for $t \geq -1$.

83. [Walther Janous] Let $n > 2$ and let $x_1, x_2, \ldots, x_n > 0$ add up to 1. Prove that

$$\prod_{i=1}^{n}\left(1+\frac{1}{x_i}\right) \geq \prod_{i=1}^{n}\left(\frac{n-x_i}{1-x_i}\right).$$

Crux Mathematicorum

First solution:

The most natural idea is to use the fact that

$$\frac{n-x_i}{1-x_i} = 1+\frac{n-1}{x_1+x_2+\cdots+x_{i-1}+x_{i+1}+\cdots+x_n}.$$

Thus, we have

$$\prod_{i=1}^{n}\left(\frac{n-x_i}{1-x_i}\right) \leq \prod_{i=1}^{n}\left(1+\frac{1}{\sqrt[n-1]{x_1x_2\ldots x_{i-1}x_{i+1}\ldots x_n}}\right)$$

and we have to prove the inequality

$$\prod_{i=1}^{n}\left(1+\frac{1}{x_i}\right) \geq \prod_{i=1}^{n}\left(1+\frac{1}{\sqrt[n-1]{x_1x_2\ldots x_{i-1}x_{i+1}\ldots x_n}}\right).$$

But this one is not very hard, because it follows immediately by multiplying the inequalities

$$\prod_{j\neq i}\left(1+\frac{1}{x_j}\right)\geq\left(1+\sqrt[n-1]{\prod_{j\neq i}\frac{1}{x_j}}\right)^{n-1}$$

obtained from **Huygens Inequality**.

***Second solution*:**

We will prove even more, that

$$\prod_{i=1}^{n}\left(1+\frac{1}{x_i}\right)\geq\left(\frac{n^2-1}{n}\right)^n\cdot\prod_{i=1}^{n}\frac{1}{1-x_i}.$$

It is clear that this inequality is stronger than the initial one. First, let us prove that

$$\prod_{i=1}^{n}\frac{1+x_i}{1-x_i}\geq\left(\frac{n+1}{n-1}\right)^n.$$

This follows from **Jensen's Inequality** for the convex function $f(x)=\ln(1+x)-\ln(1-x)$. So, it suffices to prove that

$$\frac{\left(\frac{n+1}{n-1}\right)^n}{\prod_{i=1}^{n}x_i}\cdot\prod_{i=1}^{n}(1-x_i)^2\geq\left(\frac{n^2-1}{n}\right)^n.$$

But a quick look shows that this is exactly the inequality proved in the solution of the problem 121.

84. [Vasile Cîrtoaje, Gheorghe Eckstein] Consider positive real numbers $x_1, x_2, ..., x_n$ such that $x_1x_2...x_n = 1$. Prove that

$$\frac{1}{n-1+x_1}+\frac{1}{n-1+x_2}+...+\frac{1}{n-1+x_n}\leq 1.$$

TST 1999, Romania

***First solution*:**

Suppose the inequality is false for a certain system of n numbers. Then we can find a number $k>1$ and n numbers which add up to 1, let them be a_i, such that $\frac{1}{n-1+x_i}=ka_i$. Then we have

$$1=\prod_{i=1}^{n}\left(\frac{1}{ka_i}-n+1\right)<\prod_{i=1}^{n}\left(\frac{1}{a_i}-n+1\right).$$

We have used here the fact that $a_i < \dfrac{1}{n-1}$. Now, we write $1-(n-1)a_k = b_k$ and we find that $\sum_{k=1}^{n} b_k = 1$ and also $\prod_{k=1}^{n}(1-b_k) < (n-1)^n b_1 \dots b_n$. But this contradicts the fact that for each j we have

$$1-b_j = b_1+\dots+b_{j-1}+b_{j+1}+\dots+b_n \geq (n-1)\sqrt[n-1]{b_1\dots b_{j-1}b_{j+1}\dots b_n}.$$

***Second solution*:**

Let us write the inequality in the form

$$\frac{x_1}{n-1+x_1}+\frac{x_2}{n-1+x_2}+\ldots+\frac{x_n}{n-1+x_n}\geq 1.$$

This inequality follows by summing the following inequalities

$$\frac{x_1}{n-1+x_1}\geq\frac{x_1^{1-\frac{1}{n}}}{x_1^{1-\frac{1}{n}}+x_2^{1-\frac{1}{n}}+\ldots+x_n^{1-\frac{1}{n}}},\ldots,\frac{x_n}{n-1+x_n}\geq\frac{x_n^{1-\frac{1}{n}}}{x_1^{1-\frac{1}{n}}+x_2^{1-\frac{1}{n}}+\ldots+x_n^{1-\frac{1}{n}}},$$

The first from these inequalities is equivalent to

$$x_1^{1-\frac{1}{n}}+x_2^{1-\frac{1}{n}}+\ldots+x_n^{1-\frac{1}{n}}\geq(n-1)x_1^{-\frac{1}{n}}$$

and follows from the **AM-GM Inequality**.

Remark.

Replacing the numbers $x_1, x_2, \dots, x_n$ with $\dfrac{1}{x_1}, \dfrac{1}{x_2}, \dots, \dfrac{1}{x_n}$ respectively, the inequality becomes as follows

$$\frac{1}{1+(n-1)x_1}+\frac{1}{1+(n-1)x_2}+\ldots+\frac{1}{1+(n-1)x_n}\geq 1.$$

85. [Titu Andreescu] Prove that for any nonnegative real numbers a, b, c such that $a^2+b^2+c^2+abc=4$ we have $0 \leq ab+bc+ca-abc \leq 2$.

USAMO, 2001

***First solution (by Richard Stong)*:**

The lower bound is not difficult. Indeed, we have $ab+bc+ca \geq 3\sqrt[3]{a^2b^2c^2}$ and thus it is enough to prove that $abc \leq 3\sqrt[3]{a^2b^2c^2}$, which follows from the fact that $abc \leq 4$. The upper bound instead is hard. Let us first observe that there are two numbers among the three ones, which are both greater or equal than 1 or smaller than or equal to 1. Let them be b and c. Then we have

$$4 \geq 2bc + a^2 + abc \Rightarrow (2-a)(2+a) \geq bc(2+a) \Rightarrow bc \leq 2-a.$$

Thus, $ab+bc+ca-abc \leq ab+2-a+ac-abc$ and it is enough to prove the inequality $ab+2-a+ac-abc \leq 2 \Leftrightarrow b+c-bc \leq 1 \Leftrightarrow (b-1)(c-1) \geq 0$, which is true due to our choice.

***Second solution*:**

We won't prove again the lower part, since this is an easy problem. Let us concentrate on the upper bound. Let $a \geq b \geq c$ and let $a = x + y$, $b = x - y$. The hypothesis becomes $x^2(2 + c) + y^2(2 - c) = 4 - c^2$ and we have to prove that $(x^2 - y^2)(1 - c) \leq 2(1 - xc)$. Since $y^2 = 2 + c - \dfrac{2+c}{2-c}x^2$, the problem asks to prove the inequality $\dfrac{4x^2 - (4 - c^2)}{2 - c}(1 - c) \leq 2(1 - cx)$. Of course, we have $c \leq 1$ and $0 \leq y^2 = 2 + c - \frac{2+c}{2-c}x^2 \Rightarrow x^2 \leq 2 - c \Rightarrow x \leq \sqrt{2-c}$ (we have used the fact that $a \geq b \Rightarrow y \geq 0$ and $b \geq 0 \Rightarrow x \geq y \geq 0$). Now, consider the function $f : [0, \sqrt{2-c}] \to R$, $f(x) = 2(1 - cx) - \dfrac{4x^2 - (4 - c^2)}{2 - c}(1 - c)$. We have $f'(x) = -2c - 8x \cdot \dfrac{1 - c}{2 - c} \leq 0$ and thus f is decreasing and $f(x) \geq f(\sqrt{2-c})$. So, we have to prove that

$$f(\sqrt{2-c}) \geq 0 \Leftrightarrow 2(1-c\sqrt{2-c}) \geq (2-c)(1-c) \Leftrightarrow 3 \geq c+2\sqrt{2-c} \Leftrightarrow (1-\sqrt{2-c})^2 \geq 0$$

clearly true. Thus, the problem is solved.

86. [Titu Andreescu] Prove that for any positive real numbers a, b, c the following inequality holds

$$\frac{a+b+c}{3} - \sqrt[3]{abc} \leq max\{(\sqrt{a} - \sqrt{b})^2, (\sqrt{b} - \sqrt{c})^2, (\sqrt{c} - \sqrt{a})^2\}.$$

TST 2000, USA

***Solution*:**

A natural idea would be to assume the contrary, which means that

$$\frac{a+b+c}{3} - \sqrt[3]{abc} \leq a + b - 2\sqrt{ab}$$
$$\frac{a+b+c}{3} - \sqrt[3]{abc} \leq b + c - 2\sqrt{bc}$$
$$\frac{a+b+c}{3} - \sqrt[3]{abc} \leq c + a - 2\sqrt{ca}.$$

Adding these inequalities, we find that

$$a + b + c - 3\sqrt[3]{abc} > 2(a + b + c - \sqrt{ab} - \sqrt{bc} - \sqrt{ca}).$$

Now, we will prove that $a+b+c-3\sqrt[3]{abc} \leq 2(a+b+c-\sqrt{ab}-\sqrt{bc}-\sqrt{ca})$ and the problem will be solved. Since the above inequality is homogeneous, we may assume that $abc = 1$. Then, it becomes $2\sqrt{ab} + 2\sqrt{bc} + 2\sqrt{ca} - a - b - c \leq 3$. Now, using **Schur's inequality**, we find that for any positive reals x,y,z we have:

$$2xy + 2yz + 2zx - x^2 - y^2 - z^2 \leq \frac{9xyz}{x+y+z} \leq 3\sqrt[3]{x^2y^2z^2}$$

All we have to do is take $x = \sqrt{a}$, $y = \sqrt{b}$, $z = \sqrt{c}$ in the above inequality.

87. [Kiran Kedlaya] Let a,b,c be positive real numbers. Prove that

$$\frac{a+\sqrt{ab}+\sqrt[3]{abc}}{3} \leq \sqrt[3]{a\cdot\frac{a+b}{2}\cdot\frac{a+b+c}{3}}.$$

***Solution (by Anh Cuong)*:**
We have that

$$a+\sqrt{ab}+\sqrt[3]{abc} \leq a+\sqrt[3]{ab\frac{a+b}{2}}+\sqrt[3]{abc}.$$

Now we will prove that

$$a+\sqrt[3]{ab\frac{a+b}{2}}+\sqrt[3]{abc} \leq \sqrt[3]{a\cdot\frac{a+b}{2}\cdot\frac{a+b+c}{3}}.$$

By the **AM-GM Inequality**, we have

$$\sqrt[3]{1\cdot\frac{2a}{a+b}\cdot\frac{3a}{a+b+c}} \leq \frac{1+\dfrac{2a}{a+b}+\dfrac{3a}{a+b+c}}{3},$$

$$\sqrt[3]{1\cdot 1\cdot\frac{3b}{a+b+c}} \leq \frac{2+\dfrac{3b}{a+b+c}}{3},$$

$$\sqrt[3]{1\cdot\frac{2b}{a+b}\cdot\frac{3c}{a+b+c}} \leq \frac{1+\dfrac{2b}{a+b}+\dfrac{3c}{a+b+c}}{3}.$$

Now, just add them up and we have the desired inequality. The equality occurs when $a=b=c$.

88. Find the greatest constant k such that for any positive integer n which is not a square, $|(1+\sqrt{n})\sin(\pi\sqrt{n})|>k$.

Vietnamese IMO Training Camp, 1995

***Solution*:**
We will prove that $\frac{\pi}{2}$ is the best constant. We must clearly have $k < \left(1+\sqrt{i^2+1}\right)|\sin\left(\pi\sqrt{i^2+1}\right)|$ for all positive integers i. Because $|\sin\left(\pi\sqrt{i^2+1}\right)| = \sin\dfrac{\pi}{i+\sqrt{i^2+1}}$, we deduce that $\dfrac{\pi}{i+\sqrt{i^2+1}} \geq \sin\dfrac{\pi}{i+\sqrt{i^2+1}} > \dfrac{k}{1+\sqrt{i^2+1}}$, from where it follows that $k \leq \frac{\pi}{2}$. Now, let us prove that this constant is good. Clearly, the inequality can be written

$$\sin\left(\pi\cdot\{\sqrt{n}\}\right) > \frac{\pi}{2(1+\sqrt{n})}.$$

We have two cases

i) The first case is when $\{\sqrt{n}\} \le \frac{1}{2}$. Of course,

$$\{\sqrt{n}\} \ge \sqrt{n} - \sqrt{n-1} = \frac{1}{\sqrt{n}+\sqrt{n-1}},$$

and because $\sin x \ge x - \frac{x^3}{6}$ we find that

$$\sin(\pi\{\sqrt{n}\}) \ge \sin\frac{\pi}{\sqrt{n-1}+\sqrt{n}} \ge \left(\frac{\pi}{\sqrt{n-1}+\sqrt{n}}\right) - \frac{1}{6}\left(\frac{\pi}{\sqrt{n-1}+\sqrt{n}}\right)^3.$$

Let us prove that the last quantity is at least $\frac{\pi}{2(1+\sqrt{n})}$. This comes down to

$$\frac{2+\sqrt{n}-\sqrt{n-1}}{1+\sqrt{n}} > \frac{\pi^2}{3\left(\sqrt{n}+\sqrt{n-1}\right)^2},$$

or $6(\sqrt{n}+\sqrt{n-1})^2 + 3(\sqrt{n}+\sqrt{n-1}) > \pi^2(1+\sqrt{n})$ and it is clear.

ii) The second case is when $\{\sqrt{n}\} > \frac{1}{2}$. Let $x = 1 - \{\sqrt{n}\} < \frac{1}{2}$ and let $n = k^2 + p, 1 \le p \le 2k$. Because $\{\sqrt{n}\} > \frac{1}{2} \Rightarrow p \ge k+1$. Then it is easy to see that $x \ge \frac{1}{k+1+\sqrt{k^2+2k}}$ and so it suffices to prove that

$$\sin\frac{\pi}{k+1+\sqrt{k^2+2k}} \ge \frac{\pi}{2(1+\sqrt{k^2+k})}.$$

Using again the inequality $\sin x \ge x - \frac{x^3}{6}$ we infer that,

$$\frac{2\sqrt{k^2+k}-\sqrt{k^2+2k}-k+1}{1+\sqrt{k^2+k}} > \frac{\pi^2}{3\left(1+k+\sqrt{k^2+2k}\right)^2}.$$

But from the **Cauchy-Schwarz Inequality** we have $2\sqrt{k^2+k}-\sqrt{k^2+2k}-k \ge 0$. Because the inequality $\left(1+k+\sqrt{k^2+2k}\right)^2 > \frac{\pi^2}{3}\left(1+\sqrt{k^2+k}\right)$ holds, this case is also solved.

89. [Dung Tran Nam] Let $x, y, z > 0$ such that $(x+y+z)^3 = 32xyz$. Find the minimum and maximum of $\frac{x^4+y^4+z^4}{(x+y+z)^4}$.

Vietnam, 2004

First solution (by Tran Nam Dung):
We may of course assume that $x+y+z=4$ and $xyz=2$. Thus, we have to find the

extremal values of $\frac{x^4+y^4+z^4}{4^4}$. Now, we have

$$\begin{aligned} x^4+y^4+z^4 &= (x^2+y^2+z^2)^2 - 2\sum x^2y^2 = \\ &= (16-2\sum xy)^2 - 2(\sum xy)^2 + 4xyz(x+y+z) = \\ &= 2a^2 - 64a + 288, \end{aligned}$$

where $a = xy+yz+zx$. Because $y+z = 4-x$ and $yz = \frac{2}{x}$, we must have $(4-x)^2 \geq \frac{8}{x}$, which implies that $3-\sqrt{5} \leq x \leq 2$. Due to symmetry, we have $x, y, z \in [3-\sqrt{5}, 2]$. This means that $(x-2)(y-2)(z-2) \leq 0$ and also

$$(x-3+\sqrt{5})(y-3+\sqrt{5})(z-3+\sqrt{5}) \geq 0.$$

Clearing paranthesis, we deduce that

$$a \in \left[5, \frac{5\sqrt{5}-1}{2}\right].$$

But because $\frac{x^4+y^4+z^4}{4^4} = \frac{(a-16)^2-112}{128}$, we find that the extremal values of the expression are $\frac{383-165\sqrt{5}}{256}, \frac{9}{128}$, attained for the triples $\left(3-\sqrt{5}, \frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$, respectively $(2,1,1)$.

Second solution:

As in the above solution, we must find the extremal values of $x^2+y^2+z^2$ when $x+y+z = 1, xyz = \frac{1}{32}$, because after that the extremal values of the expression $x^4+y^4+z^4$ can be immediately found. Let us make the substitution $x = \frac{a}{4}, y = \frac{b}{4}, z = \frac{c}{2}$, where $abc = 1, a+b+2c = 4$. Then $x^2+y^2+z^2 = \frac{a^2+b^2+4c^2}{16}$ and so we must find the extremal values of $a^2+b^2+4c^2$. Now, $a^2+b^2+4c^2 = (4-2c)^2 - \frac{2}{c} + 4c^2$ and the problem reduces to finding the maximum and minimum of $4c^2 - 8c - \frac{1}{c}$ where there are positive numbers a, b, c such that $abc = 1, a+b+2c = 4$. Of course, this comes down to $(4-2c)^2 \geq \frac{4}{c}$, or to $c \in \left[\frac{3-\sqrt{5}}{2}, 1\right]$. But this reduces to the study of the function $f(x) = 4x^2 - 8x - \frac{1}{x}$ defined for $\left[\frac{3-\sqrt{5}}{2}, 1\right]$, which is an easy task.

90. [George Tsintifas] Prove that for any $a, b, c, d > 0$,

$$(a+b)^3(b+c)^3(c+d)^3(d+a)^3 \geq 16a^2b^2c^2d^2(a+b+c+d)^4.$$

Crux Mathematicorum

Solution:

Let us apply **Mac-Laurin Inequality** for

$$x = abc, y = bcd, z = cda, t = dab.$$

We will find that

$$\left(\frac{\sum abc}{4}\right)^3 \geq \frac{\sum abc \cdot bcd \cdot cda}{4} = \frac{a^2b^2c^2d^2 \sum a}{4}.$$

Thus, it is enough to prove the stronger inequality

$$(a+b)(b+c)(c+d)(d+a) \geq (a+b+c+d)(abc+bcd+cda+dab).$$

Now, let us observe that

$(a+b)(b+c)(c+d)(d+a) = (ac+bd+ad+bc)(ac+bd+ab+cd) =$
$= (ac+bd)^2 + \sum a^2(bc+bd+cd) \geq 4abcd + \sum a^2(bc+bd+cd) =$
$= (a+b+c+d)(abc+bcd+cda+dab).$

And so the problem is solved.

91. [Titu Andreescu, Gabriel Dospinescu] Find the maximum value of the expression

$$\frac{(ab)^n}{1-ab} + \frac{(bc)^n}{1-bc} + \frac{(ca)^n}{1-ca}$$

where a, b, c are nonnegative real numbers which add up to 1 and n is some positive integer.

Solution:

First, we will treat the case $n > 1$. We will prove that the maximum value is $\frac{1}{3 \cdot 4^{n-1}}$. It is clear that $ab, bc, ca \leq \frac{1}{4}$ and so

$$\frac{(ab)^n}{1-ab} + \frac{(bc)^n}{1-bc} + \frac{(ca)^n}{1-ca} \leq \frac{4}{3}\left((ab)^n + (bc)^n + (ca)^n\right).$$

Thus, we have to prove that $(ab)^n + (bc)^n + (ca)^n \leq \frac{1}{4^n}$. Let a the maximum among a, b, c. Then we have

$$\frac{1}{4^n} \geq a^n(1-a)^n = a^n(b+c)^n \geq a^nb^n + a^nc^n + na^nb^{n-1}c \geq a^nb^n + b^nc^n + c^na^n.$$

So, we have proved that in this case the maximum is at most $\frac{1}{3 \cdot 4^{n-1}}$. But for $a = b = \frac{1}{2}, c = 0$ this value is attained and this shows that the maximum value is $\frac{1}{3 \cdot 4^{n-1}}$ for $n > 1$. Now, suppose that $n = 1$. In this case we have

$$\sum \frac{ab}{1-ab} = \sum \frac{1}{1-ab} - 3.$$

Using the fact that $a+b+c=1$, it is not difficult to prove that

$$\frac{1}{1-ab}+\frac{1}{1-bc}+\frac{1}{1-ca}=\frac{3-2\sum ab+abc}{1-\sum ab+abc-a^2b^2c^2}.$$

We will prove that $\sum \frac{ab}{1-ab} \leq \frac{3}{8}$. With the above observations, this reduces to

$$\sum ab \leq \frac{3-27(abc)^2+19abc}{11}.$$

But using **Schur's Inequality** we infer that

$$\sum ab \leq \frac{1+9abc}{4}$$

and so it is enough to show that

$$\frac{9abc+1}{4} \leq \frac{3-27(abc)^2+19abc}{11} \Leftrightarrow 108(abc)^2+23abc \leq 1,$$

which is true because $abc \leq \frac{1}{27}$.

Hence for $n=1$, the maximum value is $\frac{3}{8}$ attained for $a=b=c$.

92. Let a,b,c be positive real numbers. Prove that

$$\frac{1}{a(1+b)}+\frac{1}{b(1+c)}+\frac{1}{c(1+a)} \geq \frac{3}{\sqrt[3]{abc}(1+\sqrt[3]{abc})}.$$

Solution:

The following observation is crucial

$$\begin{aligned}(1+abc)\left(\frac{1}{a(1+b)}+\frac{1}{b(1+c)}+\frac{1}{c(1+a)}\right)+3 &= \sum \frac{1+abc+a+ab}{a(1+b)} = \\ &= \sum \frac{1+a}{a(1+b)}+\sum \frac{b(c+1)}{1+b}.\end{aligned}$$

We use now twice the **AM-GM Inequality** to find that

$$\sum \frac{1+a}{a(1+b)}+\sum \frac{b(c+1)}{1+b} \geq \frac{3}{\sqrt[3]{abc}}+3\sqrt[3]{abc}.$$

And so we are left with the inequality

$$\frac{\frac{3}{\sqrt[3]{abc}}+3\sqrt[3]{abc}-3}{1+abc} \geq \frac{3}{\sqrt[3]{abc}(1+\sqrt[3]{abc})},$$

which is in fact an identity!

93. [Dung Tran Nam] Prove that for any real numbers a,b,c such that $a^2+b^2+c^2=9$,

$$2(a+b+c)-abc \leq 10.$$

Vietnam, 2002

First solution (by Gheorghe Eckstein):

Because $\max\{a,b,c\} \le 3$ and $|abc| \le 10$, it is enough to consider only the cases when $a,b,c \ge 0$ or exactly of the three numbers is negative. First, we will suppose that a,b,c are nonnegative. If $abc \ge 1$, then we are done, because

$$2(a+b+c) - abc \le 2\sqrt{3(a^2+b^2+c^2)} - 1 < 10.$$

Otherwise, we may assume that $a < 1$. In this case we have

$$2(a+b+c) - abc \le 2\left(a + 2\sqrt{\frac{b^2+c^2}{2}}\right) = 2a + 2\sqrt{18-2a^2} \le 10.$$

Now, assume that not all three numbers are nonnegative and let $c < 0$.

Thus, the problem reduces to proving that for any nonnegative x,y,z whose sum of squares is 9 we have $4(x+y-z) + 2xyz \le 20$. But we can write this as $(x-2)^2+(y-2)^2+(z-1)^2 \ge 2xyz - 6z - 2$. Because $2xyz-6z-2 \le x(y^2+z^2)-6z-2 = -z^3+3z-2 = -(z-1)^2(z+2) \le 0$, the inequality follows.

Second solution:

Of course, we have $|a|,|b|,|c| \le 3$ and $|a+b+c|, |abc| \le 3\sqrt{3}$. Also, we may assume of course that a,b,c are non-zero and that $a \le b \le c$. If $c < 0$ then we have $2(a+b+c)-abc < -abc \le 3\sqrt{3} < 10$. Also, if $a \le b < 0 < c$ then we have $2(a+b+c) < 2c \le 6 < 10 + abc$ because $abc > 0$. If $a < 0 < b \le c$, using the **Cauchy-Schwarz Inequality** we find that $2b+2c-a \le 9$. Thus, $2(a+b+c) = 2b+2c-a+3a \le 9+3a$ and it remains to prove that $3a-1 \le abc$. But $a < 0$ and $2bc \le 9-a^2$, so that it remains to show that $\frac{9a-a^3}{2} \ge 3a-1 \Leftrightarrow (a+1)^2(a-2) \le 0$, which follows. So, we just have to threat the case $0 < a \le b \le c$. In this case we have $2b+2c+a \le 9$ and $2(a+b+c) \le 9+a$. So, we need to prove that $a \le 1 + abc$. This is clear if $a < 1$ and if $a > 1$ we have $b,c > 1$ and the inequality is again. Thus, the problem is solved.

94. [Vasile Cîrtoaje] Let a,b,c be positive real numbers. Prove that

$$\left(a+\frac{1}{b}-1\right)\left(b+\frac{1}{c}-1\right)+\left(b+\frac{1}{c}-1\right)\left(c+\frac{1}{a}-1\right)+\left(c+\frac{1}{a}-1\right)\left(a+\frac{1}{b}-1\right) \ge 3.$$

First solution:

With the notations $x = a+\frac{1}{b}-1$, $y = b+\frac{1}{c}-1$, $z = c+\frac{1}{a}-1$, the inequality becomes

$$xy+yz+zx \ge 3.$$

We consider without loss of generality that $x = \max\{x,y,z\}$. From

$$(x+1)(y+1)(z+1) = abc+\frac{1}{abc}+a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ge 2+a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = 5+x+y+z,$$

we get

$$xyz + xy + yz + zx \geq 4,$$

with equality if and only if $abc = 1$. Because $y+z = \frac{1}{a}+b+\frac{(c-1)^2}{c} > 0$ we distinguish two cases a) x> 0, $yz \leq 0$; b)x> 0,y> 0,z> 0.

a) x> 0, $yz \leq 0$. We have $xyz \leq 0$ and from $xyz + xy + yz + zx \geq 4$ we get $xy + yz + zx \geq 4 > 3$.

b) x> 0,y> 0,z> 0. We denote $xy + yz + zx = 3d^2$ with $d > 0$. From the mean inequalities we have

$$xy + yz + zx \geq 3\sqrt[3]{x^2y^2z^2},$$

from which we deduce that $xyz \leq d^3$. On the basis of this result, from the inequality $xyz + xy + yz + zx \geq 4$, we obtain $d^3 + 3d^2 \geq 4$, $(d-1)(d+2)^2 \geq 0$, $d \geq 1$ so $xy + yz + zx \geq 3$. With this the given inequality is proved. We have equality in the case $a = b = c = 1$.

***Second solution*:**

Let $u = x+1, v = y+1, w = z+1$. Then we have

$$uvw = u + v + w + abc + \frac{1}{abc} \geq u + v + w + 2.$$

Now, consider the function $f(t) = 2t^3+t^2(u+v+w)-uvw$. Because $\lim_{t\to\infty} f(t) = \infty$ and $f(1) \leq 0$, we can find a real number $r \geq 1$ such that $f(r) = 0$. Consider the numbers $m = \frac{u}{r}, n = \frac{v}{r}, p = \frac{w}{r}$. They verify $mnp = m+n+p+2$ we deduce from problem 49 that $mn+np+pm \geq 2(m+n+p) \Rightarrow uv+vw+wu \geq 2r(u+v+w) \geq 2(u+v+w)$. But because $u = x+1, v = y+1, w = z+1$, this last relation is equivalent to $xy + yz + zx \geq 3$, which is what we wanted.

95. [Gabriel Dospinescu] Let n be an integer greater than 2. Find the greatest real number m_n and the least real number M_n such that for any positive real numbers $x_1, x_2, \ldots, x_n$ (with $x_n = x_0, x_{n+1} = x_1$),

$$m_n \leq \sum_{i=1}^{n} \frac{x_i}{x_{i-1} + 2(n-1)x_i + x_{i+1}} \leq M_n.$$

***Solution*:**

We will prove that $m_n = \frac{1}{2(n-1)}$, $M_n = \frac{1}{2}$. First, let us see that the inequality

$$\sum_{i=1}^{n} \frac{x_i}{x_{i-1} + 2(n-1)x_i + x_{i+1}} \geq \frac{1}{2(n-1)}$$

is trivial, because $x_{i-1} + 2(n-1)x_i + x_{i+1} \le 2(n-1) \cdot \sum_{k=1}^{n} x_k$ for all i. This shows that $m_n \ge \frac{1}{2(n-1)}$. Taking $x_i = x^i$, the expression becomes

$$\frac{1}{x + x^{n-1} + 2(n-1)} + \frac{(n-2)x}{1 + 2(n-1)x + x^2} + \frac{x^{n-1}}{1 + 2(n-1)x^{n-1} + x^{n-2}}$$

and taking the limit when x approaches 0, we find that $m_n \le \frac{1}{2(n-1)}$ and thus $m_n = \frac{1}{2(n-1)}$.

Now, we will prove that $M_n \ge \frac{1}{2}$. Of course, it suffices to prove that for any $x_1, x_2, \ldots, x_n > 0$ we have

$$\sum_{i=1}^{n} \frac{x_i}{x_{i-1} + 2(n-1)x_i + x_{i+1}} \le \frac{1}{2}.$$

But it is clear that

$$\sum_{i=1}^{n} \frac{2x_i}{x_{i-1} + 2(n-1)x_i + x_{i+1}} \le \sum_{i=1}^{n} \frac{2x_i}{2\sqrt{x_{i-1} \cdot x_{i+1}} + 2(n-1)x_i} =$$

$$= \sum_{i=1}^{n} \frac{1}{n - 1 + \frac{\sqrt{x_{i-1} \cdot x_{i+1}}}{x_i}}.$$

Taking $\frac{\sqrt{x_{i-1} \cdot x_{i+1}}}{x_i} = a_i$, we have to prove that if $\prod_{i=1}^{n} a_i = 1$ then $\sum_{i=1}^{n} \frac{1}{n-1+a_i} \le 1$. But this has already been proved in the problem 84. Thus, $M_n \ge \frac{1}{2}$ and because for $x_1 = x_2 = \cdots = x_n$ we have equality, we deduce that $M_n = \frac{1}{2}$, which solves the problem.

96. [Vasile Cîrtoaje] If x, y, z are positive real numbers, then

$$\frac{1}{x^2 + xy + y^2} + \frac{1}{y^2 + yz + z^2} + \frac{1}{z^2 + zx + z^2} \ge \frac{9}{(x+y+z)^2}.$$

Gazeta Matematică

Solution:

Considering the relation

$$x^2 + xy + y^2 = (x+y+z)^2 - (xy + yz + zx) - (x+y+z)z,$$

we get

$$\frac{(x+y+z)^2}{x^2+xy+y^2} = \frac{1}{1-\dfrac{xy+yz+zx}{(x+y+z)^2} - \dfrac{z}{x+y+z}},$$

or

$$\frac{(x+y+z)^2}{x^2+xy+y^2} = \frac{1}{1-(ab+bc+ca)-c},$$

where $a = \dfrac{x}{x+y+z}$, $b = \dfrac{y}{x+y+z}$, $c = \dfrac{z}{x+y+z}$. The inequality can be rewritten as

$$\frac{1}{1-d-c} + \frac{1}{1-d-b} + \frac{1}{1-d-a} \geq 9,$$

where a, b, c are positive reals with $a+b+c=1$ and $d = ab+bc+ca$. After making some computations the inequality becomes

$$9d^3 - 6d^2 - 3d + 1 + 9abc \geq 0$$

or

$$d(3d-1)^2 + (1-4d+9abc) \geq 0.$$

which is **Schur's Inequality**.

97. [Vasile Cîrtoaje] For any $a, b, c, d > 0$ prove that

$$2(a^3+1)(b^3+1)(c^3+1)(d^3+1) \geq (1+abcd)(1+a^2)(1+b^2)(1+c^2)(1+d^2).$$

Gazeta Matematică

***Solution*:**

Using **Huygens Inequality**

$$\prod(1+a^4) \geq (1+abcd)^4,$$

we notice that it is enough to show that that

$$2^4 \prod(a^3+1)^4 \geq \prod(1+a^4)(1+a^2)^4.$$

Of course, it suffices to prove that $2(a^3+1)^4 \geq (a^4+1)(a^2+1)^4$ for any positive real a. But $(a^2+1)^4 \leq (a+1)^2(a^3+1)^2$ and we are left with the inequality $2(a^3+1)^2 \geq (a+1)^2(a^4+1) \;\Leftrightarrow\; 2(a^2-a+1)^2 \geq a^4+1 \;\Leftrightarrow\; (a-1)^4 \geq 0$, which follows.

98. Prove that for any real numbers a, b, c,

$$(a+b)^4 + (b+c)^4 + (c+a)^4 \geq \frac{4}{7}(a^4+b^4+c^4).$$

Vietnam TST, 1996

Solution:

Let us make the substitution $a+b=2z, b+c=2x, c+a=2y$. The inequality becomes $\sum(y+z-x)^4 \le 28\sum x^4$. Now, we have the following chain of identities

$$\sum(y+z-x)^4 = \sum\left(\sum x^2 + 2yz - 2xy - 2xz\right)^2 = 3\left(\sum x^2\right)^2 + 4\left(\sum x^2\right)$$
$$\left(\sum(yz-xy-xz)\right) + 4\sum(xy+xz-yz)^2 = 3\left(\sum x^2\right)^2 - 4\left(\sum xy\right)\left(\sum x^2\right) +$$
$$+16\sum x^2y^2 - 4\left(\sum xy\right)^2 = 4\left(\sum x^2\right)^2 + 16\sum x^2y^2 - \left(\sum x\right)^4 \le 28\sum x^4$$

because $\left(\sum x^2\right)^2 \le 3\sum x^4, \sum x^2y^2 \le \sum x^4$.

99. Prove that if a, b, c are positive real numbers such that $abc = 1$, then

$$\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \le \frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c}.$$

Bulgaria, 1997

Solution:

Let $x = a+b+c$ and $y = ab+bc+ca$. Using brute-force, it is easy to see that the left hand side is $\dfrac{x^2+4x+y+3}{x^2+2x+y+xy}$, while the right hand side is $\dfrac{12+4x+y}{9+4x+2y}$. Now, the inequality becomes

$$\frac{x^2+4x+y+3}{x^2+2x+y+xy} - 1 \le \frac{12+4x+y}{9+4x+2y} - 1 \ \Leftrightarrow\ \frac{2x+3-xy}{x^2+2x+y+xy} \le \frac{3-y}{9+4x+2y}.$$

For the last inequality, we clear denominators. Then using the inequalities $x \ge 3, y \ge 3, x^2 \ge 3y$, we have

$$\frac{5}{3}x^2y \ge 5x^2, \frac{x^2y}{3} \ge y^2, xy^2 \ge 9x, 5xy \ge 15x, xy \ge 3y \text{ and } x^2y \ge 27.$$

Summing up these inequalities, the desired inequality follows.

100. [Dung Tran Nam] Find the minimum value of the expression $\dfrac{1}{a} + \dfrac{2}{b} + \dfrac{3}{c}$ where a, b, c are positive real numbers such that $21ab + 2bc + 8ca \le 12$.

Vietnam, 2001

First solution (by Dung Tran Nam):

Let $\dfrac{1}{a} = x, \dfrac{2}{b} = y, \dfrac{3}{c} = z$. Then it is easy to check that the condition of the problem becomes $2xyz \ge 2x + 4y + 7z$. And we need to minimize $x+y+z$. But

$$z(2xy-7) \ge 2x+4y \Rightarrow \begin{cases} 2xy > 7 \\ z \ge \dfrac{2x+4y}{2xy-7} \end{cases}$$

Now, we transform the expression so that after one application of the **AM-GM Inequality** the numerator $3xy-7$ should vanish $x+y+z \geq x+y+\frac{2x+4y}{2xy-7} = x+\frac{11}{2x}+y-\frac{7}{2x}+\frac{2x+\frac{14}{x}}{2xy-7} \geq x+\frac{11}{2x}+2\sqrt{1+\frac{7}{x^2}}$. But, it is immediate to prove that $2\sqrt{1+\frac{7}{x^2}} \geq \frac{3+\frac{7}{x}}{2}$ and so $x+y+z \geq \frac{3}{2}+x+\frac{9}{x} \geq \frac{15}{2}$. We have equality for $x=3, y=\frac{5}{2}, z=2$. Therefore, in the initial problem the answer is $\frac{15}{2}$, achieved for $a=\frac{1}{3}, b=\frac{4}{5}, c=\frac{3}{2}$.

Second solution:

We use the same substitution and reduce the problem to finding the minimum value of $x+y+z$ when $2xyz \geq 2x+4y+7z$. Applying the weighted the **AM-GM Inequality** we find that

$$x+y+z \geq \left(\frac{5x}{2}\right)^{\frac{2}{5}}(3y)^{\frac{1}{3}}\left(\frac{15z}{4}\right)^{\frac{4}{15}}.$$

And also $2x+4y+7z \geq 10^{\frac{1}{5}} \cdot 12^{\frac{1}{3}} \cdot 5^{\frac{7}{15}} \cdot x^{\frac{1}{5}} \cdot y^{\frac{1}{3}} \cdot z^{\frac{7}{15}}$. This means that $(x+ +y+z)^2(2x+4y+7z) \geq \frac{225}{2}xyz$. Because $2xyz \geq 2x+4y+7z$, we will have

$$(x+y+z)^2 \geq \frac{225}{4} \Rightarrow x+y+z \geq \frac{15}{2}$$

with equality for $x=3, y=\frac{5}{2}, z=2$.

101. [Titu Andreescu, Gabriel Dospinescu] Prove that for any $x, y, z, a, b, c > 0$ such that $xy+yz+zx=3$,

$$\frac{a}{b+c}(y+z)+\frac{b}{c+a}(z+x)+\frac{c}{a+b}(x+y) \geq 3.$$

Solution:

We will prove the inequality

$$\frac{a}{b+c}(y+z)+\frac{b}{c+a}(z+x)+\frac{c}{a+b}(x+y) \geq \sqrt{3(xy+yz+zx)}$$

for any a, b, c, x, y, z. Because the inequality is homogeneous in x, y, z we can assume that $x+y+z=1$. But then we can apply the **Cauchy-Schwarz Inequality** so that to obtain

$$\frac{a}{b+c}x + \frac{b}{c+a}y + \frac{c}{a+b}z + \sqrt{3(xy+yz+zx)} \le$$
$$\le \sqrt{\sum\left(\frac{a}{b+c}\right)^2}\sqrt{\sum x^2} + \sqrt{\frac{3}{4}}\sqrt{\sum xy} + \sqrt{\frac{3}{4}}\sqrt{\sum xy} \le$$
$$\le \sqrt{\sum\left(\frac{a}{b+c}\right)^2 + \frac{3}{2}}\sqrt{\sum x^2 + 2\sum xy} = \sqrt{\sum\left(\frac{a}{b+c}\right)^2 + \frac{3}{2}}.$$

Thus, we are left with the inequality $\sqrt{\sum\left(\frac{a}{b+c}\right)^2 + \frac{3}{2}} \le \sum \frac{a}{b+c}$. But this one is equivalent to $\sum \frac{ab}{(c+a)(c+b)} \ge \frac{3}{4}$, which is trivial.

Remark.

A stronger inequality is the following:

$$\frac{a}{b+c}(y+z) + \frac{b}{c+a}(z+x) + \frac{c}{a+b}(x+y) \ge \sum\sqrt{(x+y)(x+z)} - (x+y+z),$$

which may be obtained by applying the **Cauchy-Schwarz Inequality**, as follows

$$\frac{a}{b+c}(y+z) + \frac{b}{c+a}(z+x) + \frac{c}{a+b}(x+y) =$$
$$(a+b+c)\left(\frac{y+z}{b+c} + \frac{z+x}{c+a} + \frac{x+y}{a+b}\right) - 2(x+y+z) \ge \frac{1}{2}(\sqrt{y+z} + \sqrt{z+x} + \sqrt{x+y})^2 -$$
$$2(x+y+z) = \sum\sqrt{(x+y)(x+z)} - (x+y+z).$$

A good exercise for readers is to show that

$$\sum\sqrt{(x+y)(x+z)} \ge x+y+z+\sqrt{3(xy+yz+zx)}.$$

102. Let a, b, c be positive real numbers. Prove that

$$\frac{(b+c-a)^2}{(b+c)^2+a^2} + \frac{(c+a-b)^2}{(c+a)^2+b^2} + \frac{(a+b-c)^2}{(a+b)^2+c^2} \ge \frac{3}{5}.$$

Japan, 1997

First solution:

Let $x = \frac{b+c}{a}, y = \frac{c+a}{b}, z = \frac{a+b}{c}$. The inequality can be written

$$\sum \frac{(x-1)^2}{x^2+1} \ge \frac{3}{5}.$$

Using the **Cauchy-Schwarz Inequality**, we find that

$$\sum \frac{(x-1)^2}{x^2+1} \ge \frac{(x+y+z-3)^2}{x^2+y^2+z^2+3}$$

and so it is enough to prove that

$$\frac{(x+y+z-3)^2}{x^2+y^2+z^2+3} \geq \frac{3}{5} \Leftrightarrow (\sum x)^2 - 15\sum x + 3\sum xy + 18 \geq 0.$$

But from **Schur's Inequality**, after some computations, we deduce that $\sum xy \geq 2\sum x$. Thus, we have

$$\left(\sum x\right)^2 - 15\sum x + 3\sum xy + 18 \geq \left(\sum x\right)^2 - 9\sum x + 18 \geq 0,$$

the last one being clearly true since $\sum x \geq 6$.

***Second solution*:**

Of course, we may take $a+b+c=2$. The inequality becomes

$$\sum \frac{4(1-a)^2}{2+2(1-a)^2} \geq \frac{3}{5} \Leftrightarrow \sum \frac{1}{1+(1-a)^2} \leq \frac{27}{10}.$$

But with the substitution $1-a=x, 1-b=y, 1-c=z$, the inequality reduces to that from problem 47.

103. [Vasile Cîrtoaje, Gabriel Dospinescu] Prove that if $a_1, a_2, \ldots, a_n \geq 0$ then

$$a_1^n + a_2^n + \cdots + a_n^n - na_1a_2\ldots a_n \geq (n-1)\left(\frac{a_1+a_2+\cdots+a_{n-1}}{n-1} - a_n\right)^n$$

where a_n is the least among the numbers $a_1, a_2, \ldots, a_n$.

***Solution*:**

Let $a_i - a_n = x_i \geq 0$ for $i \in \{1, 2, \ldots, n-1\}$. Now, let us look at

$$\sum_{i=1}^{n} a_i^n - n \cdot \prod_{i=1}^{n} a_i - (n-1)\left(\frac{\sum\limits_{i=1}^{n-1} a_i}{n-1} - a_n\right)^n$$

as a polynomial in $a = a_n$. It is in fact

$$a^n + \sum_{i=1}^{n-1}(a+x_i)^n - na\prod_{i=1}^{n-1}(a+x_i) - (n-1)\left(\frac{x_1+x_2+\cdots+x_{n-1}}{n-1}\right)^n.$$

We will prove that the coefficient of a^k is nonnegative for all $k \in \{0, 1, \ldots, n-1\}$, because clearly the degree of this polynomial is at most $n-1$. For $k=0$, this follows from the convexity of the function $f(x) = x^n$

$$\sum_{i=1}^{n-1} x_i^n \geq (n-1)\left(\frac{\sum\limits_{i=1}^{n-1} x_i}{n-1}\right)^n.$$

For $k > 0$, the coefficient of a^k is

$$\binom{n}{k}\sum_{i=1}^{n-1} x_i^{n-k} - n \sum_{1\le i_1<\cdots<i_{n-k}\le n-1} x_{i_1}x_{i_2}\dots x_{i_{n-k}}.$$

Let us prove that this is nonnegative. From the **AM-GM Inequality** we have

$$n \sum_{1\le i_1<i_2<\cdots<i_{n-k}\le n-1} x_{i_1}x_{i_2}\dots x_{i_{n-k}} \le$$

$$\frac{n}{n-k} \sum_{1\le i_1<i_2<\cdots<i_{n-k}\le n-1} \left(x_{i_1}^{n-k} + x_{i_2}^{n-k} + \cdots + x_{i_{n-k}}^{n-k}\right) = \frac{k}{n-1}\cdot\binom{n}{k}\sum_{i=1}^{n-1} x_i^{n-k}$$

which is clearly smaller than $\binom{n}{k}\sum_{i=1}^{n-1} x_i^{n-k}$. This shows that each coefficient of the polynomial is nonnegative and so this polynomial takes nonnegative values when restricted to nonnegative numbers.

104. [Turkevici] Prove that for all positive real numbers x, y, z, t,

$$x^4 + y^4 + z^4 + t^4 + 2xyzt \ge x^2y^2 + y^2z^2 + z^2t^2 + t^2x^2 + x^2z^2 + y^2t^2.$$

Kvant

***Solution*:**

Clearly, it is enough to prove the inequality if $xyzt = 1$ and so the problem becomes

If a, b, c, d have product 1, then $a^2 + b^2 + c^2 + d^2 + 2 \ge ab + bc + cd + da + ac + bd$. Let d the minimum among a, b, c, d and let $m = \sqrt[3]{abc}$. We will prove that

$$a^2 + b^2 + c^2 + d^2 + 2 - (ab + bc + cd + da + ac + bd) \ge d^2 + 3m^2 + 2 - (3m^2 + 3md),$$

which is in fact

$$a^2 + b^2 + c^2 - ab - bc - ca \ge d\left(a + b + c - 3\sqrt[3]{abc}\right).$$

Because $d \le \sqrt[3]{abc}$, proving this first inequality comes down to the inequality

$$a^2 + b^2 + c^2 - ab - bc - ca \ge \sqrt[3]{abc}\left(a + b + c - 3\sqrt[3]{abc}\right).$$

Take $u = \frac{a}{\sqrt[3]{abc}}, v = \frac{b}{\sqrt[3]{abc}}, w = \frac{c}{\sqrt[3]{abc}}$. Using problem 74, we find that

$$u^2 + v^2 + w^2 + 3 \ge u + v + w + uv + vw + wu$$

which is exactly $a^2 + b^2 + c^2 - ab - bc - ca \ge \sqrt[3]{abc}\left(a + b + c - 3\sqrt[3]{abc}\right)$. Thus, it remains to prove that $d^2 + 2 \ge 3md \Leftrightarrow d^2 + 2 \ge 3\sqrt[3]{d^2}$, which is clear.

105. Prove that for any real numbers $a_1, a_2, \ldots, a_n$ the following inequality holds

$$\left(\sum_{i=1}^{n} a_i\right)^2 \leq \sum_{i,j=1}^{n} \frac{ij}{i+j-1} a_i a_j.$$

***Solution*:**
Observe that

$$\begin{aligned}\sum_{i,j=1}^{n} \frac{ij}{i+j-1} a_i a_j &= \sum_{i,j=1}^{n} ia_i \cdot ja_j \int_0^1 t^{i+j-2} dt = \\ &= \int_0^1 \left(\sum_{i,j=1}^{n} ia_i \cdot ja_j \cdot t^{i-1+j-1}\right) dt = \\ &= \int_0^1 \left(\sum_{i=1}^{n} ia_i \cdot t^{i-1}\right)^2 .\end{aligned}$$

Now, using the **Cauchy-Schwarz Inequality for integrals**, we get

$$\int_0^1 \left(\sum_{i=1}^{n} ia_i \cdot t^{i-1}\right)^2 dt \geq \left(\int_0^1 \left(\sum_{i=1}^{n} ia_i \cdot t^{i-1}\right) dt\right)^2 = \left(\sum_{i=1}^{n} a_i\right)^2,$$

which ends the proof.

106. Prove that if $a_1, a_2, \ldots, a_n, b_1, \ldots, b_n$ are real numbers between 1001 and 2002, inclusively, such that $a_1^2 + a_2^2 + \cdots + a_n^2 = b_1^2 + b_2^2 + \cdots + b_n^2$, then we have the inequality

$$\frac{a_1^3}{b_1} + \frac{a_2^3}{b_2} + \cdots + \frac{a_n^3}{b_n} \leq \frac{17}{10}\left(a_1^2 + a_2^2 + \cdots + a_n^2\right).$$

TST Singapore

***Solution*:**
The key ideas are that $\frac{a_i}{b_i} \in \left[\frac{1}{2}, 2\right]$ for any i and that for all $x \in \left[\frac{1}{2}, 2\right]$ we have the inequality $x^2 + 1 \leq \frac{5}{2}x$. Consequently, we have

$$\frac{5}{2} \cdot \frac{a_i}{b_i} \geq 1 + \frac{a_1^2}{b_i^2} \Rightarrow \frac{5}{2} a_i b_i \geq a_i^2 + b_i^2$$

and also

$$\frac{5}{2} \sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} \left(a_i^2 + b_i^2\right) = 2 \sum_{i=1}^{n} a_i^2 \ (1)$$

Now, the observation that $\frac{a_i^2}{b_i^2} = \frac{\frac{a_i^3}{b_i}}{a_i \cdot b_i}$ and the inequality $\frac{5}{2} \cdot \frac{a_i}{b_i} \geq 1 + \frac{a_i^2}{b_i^2}$ allow us to write $\frac{5}{2} a_i^2 \geq \frac{a_i^3}{b_i} + a_i b_i$ and adding up these inequalities yields

$$\frac{5}{2} \sum_{i=1}^{n} a_i^2 \geq \sum_{i=1}^{n} \left(\frac{a_i^3}{b_i} + a_i b_i \right) = \sum_{i=1}^{n} \frac{a_i^3}{b_i} + \sum_{i=1}^{n} a_i b_i \ (2)$$

Using (1) and (2) we find that

$$\frac{a_1^3}{b_1} + \frac{a_2^3}{b_2} + \cdots + \frac{a_n^3}{b_n} \leq \frac{17}{10} \left(a_1^2 + a_2^2 + \cdots + a_n^2 \right),$$

which is the desired inequality.

107. [Titu Andreescu, Gabriel Dospinescu] Prove that if a, b, c are positive real numbers which add up to 1, then

$$(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) \geq 8(a^2 b^2 + b^2 c^2 + c^2 a^2)^2.$$

Solution:
Let $x = \frac{1}{a}, y = \frac{1}{b}, z = \frac{1}{c}$. We find the equivalent form if $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$ then

$$(x^2 + y^2)(y^2 + z^2)(z^2 + x^2) \geq 8(x^2 + y^2 + z^2)^2.$$

We will prove the following inequality

$$(x^2 + y^2)(y^2 + z^2)(z^2 + x^2) \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right)^2 \geq 8(x^2 + y^2 + z^2)^2$$

for any positive numbers x, y, z.
Write $x^2 + y^2 = 2c, y^2 + z^2 = 2a, z^2 + x^2 = 2b$. Then the inequality becomes

$$\sum \sqrt{\frac{abc}{b + c - a}} \geq \sum a.$$

Recall **Schur's Inequality**

$$\sum a^4 + abc(a + b + c) \geq \sum a^3(b + c) \Leftrightarrow abc(a + b + c) \geq \sum a^3(b + c - a).$$

Now, using **Hölder's Inequality**, we find that

$$\sum a^3(b + c - a) = \sum \frac{a^3}{\left(\frac{1}{\sqrt{b + c - a}} \right)^2} \geq \frac{\left(\sum a \right)^3}{\left(\sum \frac{1}{\sqrt{b + c - a}} \right)^2}.$$

Combining the two inequalities, we find that

$$\sum \sqrt{\frac{abc}{b + c - a}} \geq \sum a$$

and so the inequality is proved.

108. [Vasile Cîrtoaje] If a,b,c,d are positive real numbers such that $abcd=1$, then

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}+\frac{1}{(1+c)^2}+\frac{1}{(1+d)^2}\geq 1.$$

Gazeta Matematică

***Solution*:**

If follows by summing the inequalities

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}\geq\frac{1}{1+ab},$$
$$\frac{1}{(1+c)^2}+\frac{1}{(1+d)^2}\geq\frac{1}{1+cd}.$$

The first from these inequalities follows from

$$\begin{aligned}\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}-\frac{1}{1+ab} &= \frac{ab(a^2+b^2)-a^2b^2-2ab+1}{(1+a)^2(1+b)^2(1+c)^2}=\\ &= \frac{ab(a-b)^2+(ab-1)^2}{(1+a)^2(1+b)^2(1+ab)}\geq 0.\end{aligned}$$

Equality holds if $a=b=c=d=1$.

109. [Vasile Cîrtoaje] Let a,b,c be positive real numbers. Prove that

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}\geq\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}.$$

Gazeta Matematică

***Solution*:**

We have the following identities

$$\frac{a^2}{b^2+c^2}-\frac{a}{b+c}=\frac{ab(a-b)+ac(a-c)}{(b+c)(b^2+c^2)}$$
$$\frac{b^2}{c^2+a^2}-\frac{b}{c+a}=\frac{bc(b-c)+ab(b-a)}{(c+a)(c^2+a^2)}$$
$$\frac{c^2}{a^2+b^2}-\frac{c}{a+b}=\frac{ac(c-a)+bc(c-b)}{(b+a)(b^2+a^2)}.$$

Thus, we have

$$\sum\frac{a^2}{b^2+c^2}-\sum\frac{a}{b+c}=\sum\left[\frac{ab(a-b)}{(b+c)(b^2-c^2)}-\frac{ab(a-b)}{(a+c)(a^2+c^2)}\right]=$$
$$=\left(a^2+b^2+c^2+ab+bc+ca\right)\cdot\sum\frac{ab(a-b)^2}{(b+c)(c+a)(b^2+c^2)(c^2+a^2)}\geq 0.$$

110. [Gabriel Dospinescu] Let $a_1, a_2, \ldots, a_n$ be real numbers and let S be a non-empty subset of $\{1, 2, \ldots, n\}$. Prove that

$$\left(\sum_{i\in S} a_i\right)^2 \le \sum_{1\le i\le j\le n} (a_i + \ldots + a_j)^2.$$

TST 2004, Romania

First solution:

Denote $s_k = a_1 + \cdots + a_k$ for $k = \overline{1,n}$ and also $s_{n+1} = 0$. Define now

$$b_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{, otherwise} \end{cases}$$

Using **Abel's summation** we find that

$$\begin{aligned} \sum_{i\in S} a_i &= a_1b_1 + a_2b_2 + \cdots + a_nb_n = \\ &= s_1(b_1 - b_2) + s_2(b_2 - b_3) + \cdots + s_{n-1}(b_{n-1} - b_n) + s_nb_n + s_{n+1}(-b_1) \end{aligned}$$

Now, put $b_1 - b_2 = x_1, b_2 - b_3 = x_2, \ldots, b_{n-1} - b_n = x_{n-1}, b_n = x_n, x_{n+1} = -b_1$. So we have

$$\sum_{i\in S} a_i = \sum_{i=1}^{n+1} x_i s_i$$

and also $x_i \in \{-1, 0, 1\}$. Clearly, $\sum_{i=1}^{n+1} x_i = 0$. On the other hand, using Lagrange identity we find that

$$\begin{aligned} \sum_{1\le i\le j\le n} (a_i + \cdots + a_j)^2 &= \sum_{i=1}^{n} s_i^2 + \sum_{1\le i<j\le n} (s_j - s_i)^2 = \\ &= \sum_{1\le i<j\le n+1} (s_j - s_i)^2 = (n+1)\sum_{i=1}^{n+1} s_i^2 - \left(\sum_{i=1}^{n+1} s_i\right)^2. \end{aligned}$$

So we need to prove that

$$(n+1)\sum_{i=1}^{n+1} s_i^2 \ge \left(\sum_{i=1}^{n+1} s_i x_i\right)^2 + \left(\sum_{i=1}^{n+1} s_i\right)^2.$$

But it is clear that

$$\begin{aligned} \left(\sum_{i=1}^{n+1} s_i x_i\right)^2 + \left(\sum_{i=1}^{n+1} s_i\right)^2 &= \sum_{i=1}^{n+1} s_i^2(1 + x_i^2) \\ &+ 2\sum_{1\le i<j\le n+1} s_i s_j (x_i x_j + 1) \end{aligned}$$

Now, using the fact that $2s_is_j \le {s_i}^2 + s_j^2,\ 1 + x_ix_j \ge 0$, we can write

$$2 \sum_{1\le i<jn+1} s_is_j(x_ix_j+1) \le \sum_{1\le i<j\le n+1} (s_i^2+s_j^2)(1+x_ix_j) =$$
$$n \quad (s_1^2+\cdots+s_{n+1}^2) + s_1^2x_1(x_2+x_3+\cdots+x_n)+\cdots+s_n^2x_n(x_1+$$
$$+ \quad x_2+\cdots+x_{n-1}) = -s_1^2x_1^2 - \cdots - s_n^2x_n^2 + n(s_1^2+\cdots+s_n^2).$$

So,

$$\left(\sum_{i=1}^{n+1} s_ix_i\right)^2 + \left(\sum_{i=1}^{n+1} s_i\right)^2 \le \sum_{k=1}^{n+1} s_k^2(x_k^2+1) + \sum_{k=1}^{n+1} s_k^2(n-x_k^2) =$$

$= (n+1)\sum_{k+1}^{n+1} s_k^2$ and we are done.

***Second solution (by Andrei Neguţ)*:**

First, let us prove a lemma

Lemma

For any $a_1, a_2, \ldots, a_{2k+1} \in \mathbb{R}$ we have the inequality

$$\left(\sum_{i=0}^{k} a_{2i+1}\right)^2 \le \sum_{1\le i\le j\le 2k+1} (a_i+\cdots+a_j)^2.$$

Proof of the lemma

Let us take $s_k = a_1 + \cdots + a_k$. We have

$$\sum_{i=0}^{k} a_{2i+1} = s_1 + s_3 - s_2 + \cdots + s_{2k+1} - s_{2k}$$

and so the left hand side in the lemma is

$$\sum_{i=1}^{2k+1} s_i^2 + 2\sum_{0\le i<j\le k} s_{2i+1}s_{2j+1} + 2\sum_{1\le i<j\le k} s_{2i}s_{2j} - 2\sum_{\substack{0\le i\le k\\ 1\le j\le k}} s_{2i+1}s_{2j}$$

and the right hand side is just

$$(2k+1)\sum_{i=1}^{2k+1} s_i^2 - 2\sum_{1\le i<j\le 2k+1} s_is_j.$$

Thus, we need to prove that

$$2k\sum_{i=1}^{2k+1} s_i^2 \ge 4\sum_{0\le i<j\le k} s_{2i+1}s_{2j+1} + 4\sum_{1\le i<j\le k} s_{2i}s_{2j}$$

and it comes by adding up the inequalities

$$2s_{2j+1}s_{2i+1} \le s_{2i+1}^2 + s_{2j+1}^2,\ 2s_{2i}s_{2j} \le s_{2i}^2 + s_{2j}^2.$$

Now, let us turn back to the solution of the problem. Let us call a succession of a_i's a sequence and call a sequence that is missing from S a gap. We group the successive sequences from S and thus S will look like this

$$S = \{a_{i_1}, a_{i_1+1}, \dots, a_{i_1+k_1}, a_{i_2}, a_{i_2+1}, \dots, a_{i_2+k_2}, \dots, a_{i_r}, \dots, a_{i_r+k_r}\}$$

where $i_j + k_j < i_{j+1} - 1$. Thus, we write S as a sequence, followed by a gap, followed by a sequence, then a gap and so on. Now, take $s_1 = a_{i_1} + \cdots + a_{i_1+k_1}$, $s_2 = a_{i_1+k_1+1} + \cdots + a_{i_2-1}, \dots, s_{2r-1} = a_{i_r} + \cdots + a_{i_r+k_r}$

Then,

$$\left(\sum_{i\in S} a_i\right)^2 = (s_1 + s_3 + \cdots + s_{2r-1})^2 \le$$

$$\sum_{1\le i\le j\le 2r-1} (s_i + \cdots + s_j)^2 \le \sum_{1\le i\le j\le n} (a_i + \cdots + a_j)^2$$

the last inequality being clearly true, because the terms in the left hand side are among those from the right hand side.

Third solution:

We will prove the inequality using induction. For $n = 2$ and $n = 3$ it's easy. Suppose the inequality is true for all $k < n$ and let us prove it for n. If $1 \notin S$, then we just apply the inductive step for the numbers $a_2, \dots, a_n$, because the right hand side doesn't decrease. Now, suppose $1 \in S$. If $2 \in S$, then we apply the inductive step with the numbers $a_1 + a_2, a_3, \dots, a_n$. Thus, we may assume that $2 \notin S$. It is easy to see that $(a+b+c)^2 + c^2 \ge \frac{(a+b)^2}{2} \ge 2ab$ and thus we have $(a_1 + a_2 + \cdots + a_k)^2 + (a_2 + a_3 + \cdots + a_{k-1})^2 \ge 2a_1 a_k$ (*). Also, the inductive step for $a_3, \dots, a_n$ shows that

$$\left(\sum_{i\in S\setminus\{1\}}\right)^2 \le \sum_{3\le i\le j\le n} (a_i + \cdots + a_j)^2.$$

So, it suffices to show that

$$a_1^2 + 2a_1 \sum_{i\in S\setminus\{1\}} a_i \le \sum_{i=1}^{n} (a_1 + \cdots + a_i)^2 + \sum_{i=2}^{n} (a_2 + \cdots + a_i)^2$$

But this is clear from the fact that a_1^2 appears in the right hand side and by summing up the inequalities from (*).

111. [Dung Tran Nam] Let $x_1, x_2 \dots, x_{2004}$ be real numbers in the interval $[-1, 1]$ such that $x_1^3 + x_2^3 + \ldots + x_{2004}^3 = 0$. Find the maximal value of the $x_1 + x_2 + \cdots + x_{2004}$.

Solution:

Let us take $a_i = x_i^3$ and the function $f : [-1, 1] \to \mathbb{R}$, $f(x) = x^{\frac{1}{3}}$. We will prove first the following properties of f:

1. $f(x+y+1)+f(-1)\geq f(x)+f(y)$ if $-1<x,y<0$.
2. f is convex on $[-1,0]$ and concave on $[0,1]$.
3. If $x>0$ and $y<0$ and $x+y<0$ then $f(x)+f(y)\leq f(-1)+f(x+y+1)$ and if $x+y>0$ then $f(x)+f(y)\leq f(x+y)$. The proofs of these results are easy. Indeed, for the first one we make the substitution $x=-a^3, y=-b^3$ and it comes down to $1>a^3+b^3+(1-a-b)^3 \Leftrightarrow 1>(a+b)(a^2-ab+b^2)+1-3(a+b)+3(a+b)^2-(a+b)^3 \Leftrightarrow$ $\Leftrightarrow 3(a+b)(1-a)(1-b)>0$, which follows. The second statement is clear and the third one can be easily deduced in the same manner as 1.

From these arguments we deduce that if $(t_1,t_2,\dots,t_{2004})=t$ is the point where the maximum value of the function $g: A=\{x\in[-1,1]^{2004}|x_1+\cdots+x_n=0\}\rightarrow$ $\rightarrow \mathbb{R}, g(x_1,\dots,x_{2004})=\sum_{k=1}^{2004} f(x_k)$ (this maximum exists because this function is defined on a compact) is attained then we have that all positive components of t are equal to each other and all negative ones are -1. So, suppose we have k components equal to -1 and $2004-k$ components equal to a number a. Because $t_1+t_2+\cdots+t_{2004}=0$ we find that $a=\frac{k}{2004-k}$ and the value of g in this point is $(2004-k)\sqrt[3]{\frac{k}{2004-k}}-k$. Thus we have to find the maximum value of $(2004-k)\sqrt[3]{\frac{k}{2004-k}}-k$, when k is in the set $\{0,1,\dots,2004\}$. A short analysis with derivatives shows that the maximum is attained when $k=223$ and so the maximum value is $\sqrt[3]{223}\cdot\sqrt[3]{1781^2}-223$.

112. [Gabriel Dospinescu, Călin Popa] Prove that if $n\geq 2$ and $a_1,a_2,\dots,a_n$ are real numbers with product 1, then

$$a_1^2+a_2^2+\cdots+a_n^2-n\geq\frac{2n}{n-1}\cdot\sqrt[n]{n-1}(a_1+a_2+\cdots+a_n-n).$$

Solution:

We will prove the inequality by induction. For $n=2$ it is trivial. Now, suppose the inequality is true for $n-1$ numbers and let us prove it for n. First, it is easy to see that it is enough to prove it for $a_1,\dots,a_n>0$ (otherwise we replace $a_1,a_2,\dots,a_n$ with $|a_1|,|a_2|,\dots,|a_n|$, which have product 1. Yet, the right hand side increases). Now, let a_n the maximum number among $a_1,a_2,\dots,a_n$ and let G the geometric mean of $a_1,a_2,\dots,a_{n-1}$. First, we will prove that

$$a_1^2+a_2^2+\cdots+a_n^2-n-\frac{2n}{n-1}\cdot\sqrt[n]{n-1}(a_1+a_2+\cdots+a_n-n)\geq$$

$$\geq a_n^2+(n-1)G^2-n-\frac{2n}{n-1}\cdot\sqrt[n]{n-1}(a_n+(n-1)G-n)$$

which is equivalent to

$$a_1^2+a_2^2+\cdots+a_{n-1}^2-(n-1)\sqrt[n-1]{a_1^2a_2^2\dots a_{n-1}^2}\geq$$

$$\geq \frac{2n}{n-1} \cdot \sqrt[n]{n-1} \left(a_1 + a_2 + \cdots + a_{n-1} - (n-1) \sqrt[n-1]{a_1 a_2 \ldots a_{n-1}}\right).$$

Because, $\sqrt[n-1]{a_1 a_2 \ldots a_{n-1}} \leq 1$ and $a_1 + a_2 + \cdots + a_{n-1} - (n-1) \sqrt[n-1]{a_1 a_2 \ldots a_{n-1}} \geq 0$, it is enough to prove the inequality

$$a_1^2 + \cdots + a_{n-1}^2 - (n-1)G^2 \geq \frac{2n}{n-1} \cdot \sqrt[n]{n-1} \cdot G \cdot (a_1 + \cdots + a_{n-1} - (n-1)G).$$

Now, we apply the inductive hypothesis for the numbers $\frac{a_1}{G}, \ldots, \frac{a_{n-1}}{G}$ which have product 1 and we infer that

$$\frac{a_1^2 + \cdots + a_{n-1}^2}{G^2} - n + 1 \geq \frac{2(n-1)}{n-2} \cdot \sqrt[n-1]{n-2} \left(\frac{a_1 + \cdots + a_{n-1}}{G} - n + 1\right)$$

and so it suffices to prove that

$$\frac{2(n-1)}{n-2} \cdot \sqrt[n-1]{n-2}(a_1 + \cdots + a_{n-1} - (n-1)G) \geq \frac{2n}{n-1} \cdot \sqrt[n]{n-1}(a_1 + \cdots + a_{n-1} - (n-1)G),$$

which is the same as $1 + \frac{1}{n(n-2)} \geq \frac{\sqrt[n]{n-1}}{\sqrt[n-1]{n-2}}$. This becomes

$$\left(1 + \frac{1}{n(n-2)}\right)^{n(n-1)} \geq \frac{(n-1)^{n-1}}{(n-2)^n}$$

and it follows for $n > 4$ from

$$\left(1 + \frac{1}{n(n-2)}\right)^{n(n-1)} > 2$$

and

$$\frac{(n-1)^{n-1}}{(n-2)^n} = \frac{1}{n-2} \cdot \left(1 + \frac{1}{n-2}\right) \cdot \left(1 + \frac{1}{n-2}\right)^{n-2} < \frac{e}{n-2}\left(1 + \frac{1}{n-2}\right) < 2.$$

For $n = 3$ and $n = 4$ it is easy to check.

Thus, we have proved that

$$a_1^2 + a_2^2 + \cdots + a_n^2 - n - \frac{2n}{n-1} \cdot \sqrt[n]{n-1}(a_1 + a_2 + \cdots + a_n - n) \geq$$

$$\geq a_n^2 + (n-1)G^2 - n - \frac{2n}{n-1} \cdot \sqrt[n]{n-1}(a_n + (n-1)G - n)$$

and it is enough to prove that

$$x^{2(n-1)} + \frac{n-1}{x^2} - n \geq \frac{2n}{n-1} \cdot \sqrt[n]{n-1} \left(x^{n-1} + \frac{n-1}{x} - n\right)$$

for all $x \geq 1$ (we took $x = \frac{1}{G}$). Let us consider the function

$$f(x) = x^{2(n-1)} + \frac{n-1}{x^2} - n - \frac{2n}{n-1} \cdot \sqrt[n]{n-1} \left(x^{n-1} + \frac{n-1}{x} - n\right).$$

We have

$$f'(x) = 2 \cdot \frac{x^n - 1}{x^2} \cdot \left[\frac{(n-1)(x^n + 1)}{x} - n\sqrt[n]{n-1}\right] \geq 0$$

because

$$x^{n-1}+\frac{1}{x}=x^{n-1}+\frac{1}{(n-1)x}+\cdots+\frac{1}{(n-1)x}\geq n\sqrt[n]{\frac{1}{(n-1)^{n-1}}}.$$

Thus, f is increasing and so $f(x)\geq f(1)=0$. This proves the inequality.

113. [Vasile Cîrtoaje] If a,b,c are positive real numbers, then

$$\sqrt{\frac{2a}{a+b}}+\sqrt{\frac{2b}{b+c}}+\sqrt{\frac{2c}{c+a}}\leq 3.$$

Gazeta Matematică

***First solution*:**

With the notations $x=\sqrt{\frac{b}{a}}$, $y=\sqrt{\frac{c}{b}}$, $z=\sqrt{\frac{a}{c}}$ the problem reduces to proving that $xyz=1$ implies

$$\sqrt{\frac{2}{1+x^2}}+\sqrt{\frac{2}{1+y^2}}+\sqrt{\frac{2}{1+z^2}}\leq 3.$$

We presume that $x\leq y\leq z$, which implies $xy\leq 1$ and $z\geq 1$. We have

$$\left(\sqrt{\frac{2}{1+x^2}}+\sqrt{\frac{2}{1+y^2}}\right)^2\leq 2\left(\frac{2}{1+x^2}+\frac{2}{1+y^2}\right)=4\left[1+\frac{1-x^2y^2}{(1+x^2)(1+y^2)}\right]\leq$$

$$4\left[1+\frac{1-x^2y^2}{(1+xy)^2}\right]=\frac{8}{1+xy}=\frac{8z}{z+1}.$$

so

$$\sqrt{\frac{2}{1+x^2}}+\sqrt{\frac{2}{1+y^2}}\leq 2\sqrt{\frac{2z}{z+1}}$$

and we need to prove that

$$2\sqrt{\frac{2z}{z+1}}+\sqrt{\frac{2}{1+z^2}}\leq 3.$$

Because

$$\sqrt{\frac{2}{1+z^2}}\leq\frac{2}{1+z}$$

we only need to prove that

$$2\sqrt{\frac{2z}{z+1}}+\frac{2}{1+z}\leq 3.$$

This inequality is equivalent to

$$1+3z-2\sqrt{2z(1+z)}\geq 0, (\sqrt{2z}-\sqrt{z+1})^2\geq 0,$$

and we are done.

***Second solution*:**

Clearly, the problem asks to prove that if $xyz = 1$ then

$$\sum \sqrt{\frac{2}{x+1}} \le 3.$$

We have two cases. The first and easy one is when $xy + yz + zx \ge x + y + z$. In this case we can apply the **Cauchy-Schwarz Inequality** to get

$$\sum \sqrt{\frac{2}{x+1}} \le \sqrt{3 \sum \frac{2}{x+1}}.$$

But

$$\sum \frac{1}{x+1} \le \frac{3}{2} \quad \Leftrightarrow \quad \sum 2(xy + x + y + 1) \le 3(2 + x + y + z + xy + yz + zx) \Leftrightarrow$$
$$\Leftrightarrow \quad x + y + z \le xy + yz + zx$$

and so in this case the inequality is proved.

The second case is when $xy + yz + zx < x + y + z$. Thus,

$$(x-1)(y-1)(z-1) = x + y + z - xy - yz - zx > 0$$

and so exactly two of the numbers x, y, z are smaller than 1, let them be x and y. So, we must prove that if x and y are smaller than 1, then

$$\sqrt{\frac{2}{x+1}} + \sqrt{\frac{2}{y+1}} + \sqrt{\frac{2xy}{xy+1}} \le 3.$$

Using the **Cauchy-Schwarz Inequality**, we get

$$\sqrt{\frac{2}{x+1}} + \sqrt{\frac{2}{y+1}} + \sqrt{\frac{2xy}{xy+1}} \le 2\sqrt{\frac{1}{x+1} + \frac{1}{y+1}} + \sqrt{\frac{2xy}{xy+1}}$$

and so it is enough to prove that this last quantity is at most 3. But this comes down to

$$2 \cdot \frac{\frac{1}{x+1} + \frac{1}{y+1} - 1}{1 + \sqrt{\frac{1}{x+1} + \frac{1}{y+1}}} \le \frac{1 - \frac{2xy}{xy+1}}{1 + \sqrt{\frac{2xy}{xy+1}}}.$$

Because we have $\frac{1}{x+1} + \frac{1}{y+1} \ge 1$, the left hand side is at most

$$\frac{1}{x+1} + \frac{1}{y+1} - 1 = \frac{1 - xy}{(x+1)(y+1)}$$

and so we are left with the inequality

$$\frac{1 - xy}{(x+1)(y+1)} \le \frac{1 - xy}{(xy+1)\left(1 + \sqrt{\frac{2xy}{xy+1}}\right)} \Leftrightarrow xy + 1 + (xy+1)\sqrt{\frac{2xy}{xy+1}} \le$$
$$\le \quad xy + 1 + x + y \Leftrightarrow x + y \ge \sqrt{2xy(xy+1)}$$

which follows from $\sqrt{2xy(xy+1)} \le 2\sqrt{xy} \le x + y$. The problem is solved.

114. Prove the following inequality for positive real numbers x, y, z

$$(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\geq\frac{9}{4}$$

Iran, 1996

***First solution (by Iurie Boreico)*:**

With the substitution $x+y=c, y+z=a, z+x=b$, the inequality becomes after some easy computations

$$\sum\left(\frac{2}{ab}-\frac{1}{c^2}\right)(a-b)^2\geq 0.$$

Let $a\geq b\geq c$. If $2c^2\geq ab$, each term in the above expression is positive and we are done. So, let $2c^2<ab$. First, we prove that $2b^2\geq ac, 2a^2\geq bc$. Suppose that $2b^2<ac$. Then $(b+c)^2\leq 2(b^2+c^2)<a(b+c)$ and so $b+c<a$, false. Clearly, we can write the inequality like that

$$\left(\frac{2}{ac}-\frac{1}{b^2}\right)(a-c)^2+\left(\frac{2}{bc}-\frac{1}{a^2}\right)(b-c)^2\geq\left(\frac{1}{c^2}-\frac{2}{ab}\right)(a-b)^2.$$

We can immediately see that the inequality $(a-c)^2\geq(a-b)^2+(b-c)^2$ holds and thus if suffices to prove that

$$\left(\frac{2}{ac}+\frac{2}{bc}-\frac{1}{a^2}-\frac{1}{b^2}\right)(b-c)^2\geq\left(\frac{1}{b^2}+\frac{1}{c^2}-\frac{2}{ab}-\frac{2}{ac}\right)(a-b)^2.$$

But it is clear that $\frac{1}{b^2}+\frac{1}{c^2}-\frac{2}{ab}-\frac{2}{ac}<\left(\frac{1}{b}-\frac{1}{c}\right)^2$ and so the right hand side is at most $\frac{(a-b)^2(b-c)^2}{b^2c^2}$. Also, it is easy to see that

$$\frac{2}{ac}+\frac{2}{bc}-\frac{1}{a^2}-\frac{1}{b^2}\geq\frac{1}{ac}+\frac{1}{bc}>\frac{(a-b)^2}{b^2c^2},$$

which shows that the left hand side is at least $\frac{(a-b)^2(b-c)^2}{b^2c^2}$ and this ends the solution.

***Second solution*:**

Since the inequality is homogenous, we may assume that $xy+yz+zx=3$. Also, we make the substitution $x+y+z=3a$. From $(x+y+z)^2\geq 3(xy+yz+zx)$, we get $a\geq 1$. Now we write the inequality as follows

$$\frac{1}{(3a-z)^2}+\frac{1}{(3a-y)^2}+\frac{1}{(3a-x)^2}\geq\frac{3}{4},$$

$$4[(xy+3az)^2+(yz+3ax)^2+(zx+3ay)^2]\geq 3(9a-xyz)^2,$$

$$4(27a^4-18a^2+3+4axyz)\geq(9a-xyz)^2,$$

$$3(12a^2-1)(3a^2-4)+xyz(34a-xyz)\geq 0, \qquad (1)$$

$$12(3a^2-1)^2+208a^2 \geq (17a-xyz)^2. \qquad (2)$$

We have two cases.

i) Case $3a^2-4 \geq 0$. Since

$$34a-xyz = \frac{1}{9}[34(x+y+z)(xy+yz+zx)-9xyz] > 0,$$

the inequality (1) is true.

ii) Case $3a^2-4 < 0$. From **Schur's Inequality**

$$(x+y+z)^3-4(x+y+z)(xy+yz+zx)+9xyz \geq 0,$$

it follows that $3a^3-4a+xyz \geq 0$. Thus,

$$12(3a^2-1)^2+208a^2-(17a-xyz)^2 \geq 12(3a^2-1)^2+208a^2-a^2(3a^2+13)^2 =$$

$$= 3(4-11a^2+10a^4-3a^6) = 3(1-a^2)^2(4-3a^2)^2 \geq 0.$$

115. Prove that for any x, y in the interval $[0,1]$,

$$\sqrt{1+x^2}+\sqrt{1+y^2}+\sqrt{(1-x)^2+(1-y)^2} \geq (1+\sqrt{5})(1-xy).$$

Solution (by Faruk F. Abi-Khuzam and Roy Barbara - "A sharp inequality and the iradius conjecture"):

Let the function $F : [0,1]^2 \to \mathbb{R}, F(x,y) = \sqrt{1+x^2}+\sqrt{1+y^2}+\sqrt{(1-x)^2+(1-y)^2}-(1+\sqrt{5})(1-xy)$. It is clear that F is symmetric in x and y and also the convexity of the function $x \to \sqrt{1+x^2}$ shows that $F(x,0) \geq 0$ for all x. Now, suppose we fix y and consider F as a function in x. It's derivatives are

$$f'(x) = (1+\sqrt{5})y + \frac{x}{\sqrt{x^2+1}} - \frac{1-x}{\sqrt{(1-x)^2+(1-y)^2}}$$

and

$$f''(x) = \frac{1}{\sqrt{(1+x^2)^3}} + \frac{(1-y)^2}{\sqrt{((1-x)^2+(1-y)^2)^3}}.$$

Thus, f is convex and its derivative is increasing. Now, let

$$r = \sqrt{1-\frac{6\sqrt{3}}{(1+\sqrt{5})^2}}, c = 1+\sqrt{5}.$$

The first case we will discuss is $y \geq \frac{1}{4}$. It is easy to see that in this case we have $cy > \frac{1}{\sqrt{y^2-2y+2}}$ (the derivative of the function $y \to y^2c^2(y^2-2y+2)-1$ is positive) and so $f'(0) > 0$. Because f' is increasing, we have $f'(x) > 0$ and so f is increasing with $f(0) = F(0,y) = F(y,0) \geq 0$. Thus, in this case the inequality is proved.

Due to case 1 and to symmetry, it remains to show that the inequality holds in the cases $x \in \left[0, \frac{1}{4}\right], y \in [0, r]$ and $x \in \left[r, \frac{1}{4}\right], y \in \left[r, \frac{1}{4}\right]$.

In the first case we deduce immediately that

$$f'(x) \le r(1+\sqrt{5}) + \frac{x}{\sqrt{1+x^2}} - \frac{1-x}{\sqrt{1+(x-1)^2}}$$

Thus, we have $f'\left(\frac{1}{4}\right) < 0$, which shows that f is decreasing on $\left[0, \frac{1}{4}\right]$. Because from the first case discussed we have $f\left(\frac{1}{4}\right) \ge 0$, we will have $F(x,y) = f(x) \ge 0$ for all points (x,y) with $x \in \left[0, \frac{1}{4}\right], y \in [0, r]$.

Now, let us discuss the most important case, when $x \in \left[r, \frac{1}{4}\right], y \in \left[r, \frac{1}{4}\right]$. Let the points $O(0,0), A(1,0), B(1,1), C(0,1), M(1,y), N(x,1)$. The triangle OMN has perimeter

$$\sqrt{1+x^2} + \sqrt{1+y^2} + \sqrt{(1-x)^2+(1-y)^2} \text{ and area } \frac{1-xy}{2}.$$

But it is trivial to show that in any triangle with perimeter P and area S we have the inequality $S \le \frac{P^2}{12\sqrt{3}}$. Thus, we find that $\sqrt{1+x^2}+\sqrt{1+y^2}+\sqrt{(1-x)^2+(1-y)^2} \ge \sqrt{6\sqrt{3}}\sqrt{1-xy} \ge (1+\sqrt{5})(1-xy)$ due to the fact that $xy \ge r^2$. The proof is complete.

116. [Suranyi] Prove that for any positive real numbers $a_1, a_2, \dots, a_n$ the following inequality holds

$$(n-1)(a_1^n+a_2^n+\cdots+a_n^n)+na_1a_2\dots a_n \ge (a_1+a_2+\cdots+a_n)(a_1^{n-1}+a_2^{n-1}+\cdots+a_n^{n-1}).$$

Miklos Schweitzer Competition

Solution:

Again, we will use induction to prove this inequality, but the proof of the inductive step will be again highly non-trivial. Indeed, suppose the inequality is true for n numbers and let us prove it for $n+1$ numbers.

Due to the symmetry and homogeneity of the inequality, it is enough to prove it under the conditions $a_1 \ge a_2 \ge \cdots \ge a_{n+1}$ and $a_1 + a_2 + \cdots + a_n = 1$. We have to prove that

$$n\sum_{i=1}^{n} a_i^{n+1} + na_{n+1}^{n+1} + na_{n+1} \cdot \prod_{i=1}^{n} a_i + a_{n+1} \cdot \prod_{i=1}^{n} a_i - (1+a_{n+1})\left(\sum_{i=1}^{n} a_i^n + a_{n+1}^n\right) \ge 0.$$

But from the inductive hypothesis we have

$$(n-1)(a_1^n + a_2^n + \cdots + a_n^n) + na_1a_2\dots a_n \ge a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1}$$

and so

$$na_{n+1}\prod_{i=1}^{n}a_i \geq a_{n+1}\sum_{i=1}^{n}a_i^{n-1}-(n-1)a_{n+1}\sum_{i=1}^{n}a_i^{n}.$$

Using this last inequality, it remains to prove that

$$\left(n\sum_{i=1}^{n}a_i^{n+1}-\sum_{i=1}^{n}a_i^{n}\right)-a_{n+1}\left(n\sum_{i=1}^{n}a_i^{n}-\sum_{i=1}^{n}a_i^{n-1}\right)+$$

$$+a_{n+1}\left(\prod_{i=1}^{n}a_i+(n-1)a_{n+1}^{n}-a_{n+1}^{n-1}\right)\geq 0.$$

Now, we will break this inequality into

$$a_{n+1}\left(\prod_{i=1}^{n}a_i+(n-1)a_{n+1}^{n}-a_{n+1}^{n-1}\right)\geq 0$$

and

$$\left(n\sum_{i=1}^{n}a_i^{n+1}-\sum_{i=1}^{n}a_i^{n}\right)-a_{n+1}\left(n\sum_{i=1}^{n}a_i^{n}-\sum_{i=1}^{n}a_i^{n-1}\right)\geq 0.$$

Let us justify these two inequalities. The first one is pretty obvious

$$\prod_{i=1}^{n}a_i+(n-1)a_{n+1}^{n}-a_{n+1}^{n-1}=\prod_{i=1}^{n}(a_i-a_{n+1}+a_{n+1})+(n-1)a_{n+1}^{n}-a_{n+1}^{n-1}\geq$$

$$\geq a_{n+1}^{n}+a_{n+1}^{n-1}\cdot\sum_{i=1}^{n}(a_i-a_{n+1})+(n-1)a_{n+1}^{n}-a_{n+1}^{n-1}=0.$$

Now, let us prove the second inequality. It can be written as

$$n\sum_{i=1}^{n}a_i^{n+1}-\sum_{i=1}^{n}a_i^{n}\geq a_{n+1}\left(n\sum_{i=1}^{n}a_i^{n}-\sum_{i=1}^{n}a_i^{n-1}\right).$$

Because $n\sum_{i=1}^{n}a_i^{n}-\sum_{i=1}^{n}a_i^{n-1}\geq 0$ (using **Chebyshev's Inequality**) and $a_{n+1}\leq\frac{1}{n}$, it is enough to prove that

$$n\sum_{i=1}^{n}a_i^{n+1}-\sum_{i=1}^{n}a_i^{n}\geq\frac{1}{n}\left(n\sum_{i=1}^{n}a_i^{n}-\sum_{i=1}^{n}a_i^{n-1}\right).$$

but this one follows, because $na_i^{n+1}+\frac{1}{n}a_i^{n-1}\geq a_i^{n}$ for all i. Thus, the inductive step is proved.

117. Prove that for any $x_1,x_2,\ldots,x_n>0$ with product 1,

$$\sum_{1\leq i<j\leq n}(x_i-x_j)^2\geq\sum_{i=1}^{n}x_i^2-n.$$

A generalization of Turkevici's inequality

Solution:

Of course, the inequality can be written in the following form

$$n \geq -(n-1)\left(\sum_{k=1}^{n} x_k^2\right) + \left(\sum_{k=1}^{n} x_k\right)^2.$$

We will prove this inequality by induction. The case $n = 2$ is trivial. Suppose the inequality is true for $n-1$ and let us prove it for n. Let

$$f(x_1, x_2, \ldots, x_n) = -(n-1)\left(\sum_{k=1}^{n} x_k^2\right) + \left(\sum_{k=1}^{n} x_k\right)^2$$

and let $G = \sqrt[n-1]{x_2 x_3 \ldots x_n}$, where we have already chosen $x_1 = \min\{x_1, x_2, \ldots, x_n\}$. It is easy to see that the inequality $f(x_1, x_2, \ldots, x_n) \leq f(x_1, G, G, \ldots, G)$ is equivalent to

$$(n-1)\sum_{k=2}^{n} x_k^2 - \left(\sum_{k=2}^{n} x_k\right)^2 \geq 2x_1\left(\sum_{k=2}^{n} x_k - (n-1)G\right).$$

Clearly, we have $x_1 \leq G$ and $\sum_{k=2}^{n} x_k \geq (n-1)G$, so it suffices to prove that

$$(n-1)\sum_{k=2}^{n} x_k^2 - \left(\sum_{k=2}^{n} x_k\right)^2 \geq 2G\left(\sum_{k=2}^{n} x_k - (n-1)G\right).$$

We will prove that this inequality holds for all $x_2, \ldots, x_n > 0$. Because the inequality is homogeneous, it is enough to prove it when $G = 1$. In this case, from the induction step we already have

$$(n-2)\sum_{k=2}^{n} x_k^2 + n - 1 \geq \left(\sum_{k=2}^{n} x_k\right)^2$$

and so it suffices to prove that

$$\sum_{k=2}^{n} x_k^2 + n - 1 \geq \sum_{k=2}^{n} 2x_k \Leftrightarrow \sum_{k=2}^{n} (x_k - 1)^2 \geq 0,$$

clearly true.

Thus, we have proved that $f(x_1, x_2, \ldots, x_n) \leq f(x_1, G, G, \ldots, G)$. Now, to complete the inductive step, we will prove that $f(x_1, G, G, \ldots, G) \leq 0$. Because clearly $x_1 = \frac{1}{G^{n-1}}$, the last assertion reduces to proving that

$$(n-1)\left((n-1)G^2 + \frac{1}{G^{2(n-1)}}\right) + n \geq \left((n-1)G + \frac{1}{G^{n-1}}\right)^2$$

which comes down to

$$\frac{n-2}{G^{2n-2}} + n \geq \frac{2n-2}{G^{n-2}}$$

and this one is an immediate consequence of the **AM-GM Inequality**.

118. [Gabriel Dospinescu] Find the minimum value of the expression

$$\sum_{i=1}^{n}\sqrt{\frac{a_1a_2\ldots a_n}{1-(n-1)a_i}}$$

where $a_1, a_2, \ldots, a_n < \dfrac{1}{n-1}$ add up to 1 and $n > 2$ is an integer.

Solution:

We will prove that the minimal value is $\dfrac{1}{\sqrt{n^{n-3}}}$. Indeed, using **Suranyi's Inequality**, we find that

$$(n-1)\sum_{i=1}^{n}a_i^n + na_1a_2\ldots a_n \geq \sum_{i=1}^{n}a_i^{n-1} \Rightarrow na_1a_2\ldots a_n \geq \sum_{i=1}^{n}a_i^{n-1}(1-(n-1)a_i).$$

Now, let us observe that

$$\sum_{k=1}^{n}a_k^{n-1}(1-(n-1)a_k) = \sum_{k=1}^{n}\frac{\left(\sqrt[3]{a_k^{n-1}}\right)^3}{\left(\sqrt{\dfrac{1}{1-(n-1)a_k}}\right)^2}$$

and so, by an immediate application of **Hölder Inequality**, we have

$$\sum_{i=1}^{n}a_i^{n-1}(1-(n-1)a_i) \geq \frac{\left(\sum\limits_{k=1}^{n}a_k^{\frac{n-1}{3}}\right)^3}{\left(\sum\limits_{k=1}^{n}\sqrt{\dfrac{1}{1-(n-1)a_k}}\right)^2}.$$

But for $n > 3$, we can apply **Jensen's Inequality** to deduce that

$$\frac{\sum\limits_{k=1}^{n}a_k^{\frac{n-1}{3}}}{n} \geq \left(\frac{\sum\limits_{k=1}^{n}a_k}{n}\right)^{\frac{n-1}{3}} = \frac{1}{n^{\frac{n-1}{3}}}.$$

Thus, combining all these inequalities, we have proved the inequality for $n > 3$. For $n = 3$ it reduces to proving that

$$\sum\sqrt{\frac{abc}{b+c-a}} \geq \sum a$$

which was proved in the solution of the problem 107.

119. [Vasile Cîrtoaje] Let $a_1, a_2, \ldots, a_n < 1$ be nonnegative real numbers such that

$$a = \sqrt{\frac{a_1^2 + a_2^2 + \ldots + a_n^2}{n}} \geq \frac{\sqrt{3}}{3}.$$

Prove that
$$\frac{a_1}{1-a_1^2}+\frac{a_2}{1-a_2^2}+\ldots+\frac{a_n}{1-a_n^2}\geq\frac{na}{1-a^2}.$$

Solution:

We proceed by induction. Clearly, the inequality is trivial for n= 1. Now we suppose that the inequality is valid for $n=k-1$, $k\geq 2$ and will prove that
$$\frac{a_1}{1-a_1^2}+\frac{a_2}{1-a_2^2}+\ldots+\frac{a_k}{1-a_k^2}\geq\frac{ka}{1-a^2},$$
for
$$a=\sqrt{\frac{a_1^2+a_2^2+\ldots+a_k^2}{k}}\geq\frac{\sqrt{3}}{3}.$$
We assume, without loss of generality, that $a_1\geq a_2\geq\ldots\geq a_k$, therefore $a\geq a_k$. Using the notation
$$x=\sqrt{\frac{a_1^2+a^2+\cdots+a_{k-1}^2}{k-1}}=\sqrt{\frac{ka^2-a_k^2}{k-1}},$$
from $a\geq a_k$ it follows that $x\geq a\geq\dfrac{\sqrt{3}}{3}$. Thus, by the inductive hypothesis, we have
$$\frac{a_1}{1-a_1^2}+\frac{a_2}{1-a_2^2}+\ldots+\frac{a_{k-1}}{1-a_{k-1}^2}\geq\frac{(k-1)x}{1-x^2},$$
and it remains to show that
$$\frac{a_k}{1-a_k^2}+\frac{(k-1)x}{1-x^2}\geq\frac{ka}{1-a^2}.$$
From
$$x^2-a^2=\frac{ka^2-a_k^2}{k-1}-a^2=\frac{a^2-a_k^2}{k-1},$$
we obtain
$$(k-1)(x-a)=\frac{(a-a_k)(a+a_k)}{x+a}$$
and
$$\frac{a_k}{1-a_k^2}+\frac{(k-1)x}{1-x^2}-\frac{ka}{1-a^2}=\left(\frac{a_k}{1-a_k^2}-\frac{a}{1-a^2}\right)+(k-1)\left(\frac{x}{1-x^2}-\frac{a}{1-a^2}\right)=$$
$$\frac{-(a-a_k)(1+aa_k)}{(1-a_k^2)(1-a^2)}+\frac{(k-1)(x-a)(1+ax)}{(1-x^2)(1-a^2)}=\frac{a-a_k}{1-a^2}\left[\frac{-(1+aa_k)}{1-a_k^2}+\frac{(a+a_k)(1+ax)}{(1-x^2)(x+a)}\right]=$$
$$=\frac{(a-a_k)(x-a_k)[-1+x^2+a_k^2+xa_k+a(x+a_k)+a^2+axa_k(x+a_k)+a^2xa_k]}{(1-a^2)(1-a_k^2)(1-x^2)(x+a)}.$$
Since $x^2-a_k^2=\dfrac{ka^2-a_k^2}{k-1}-a_k^2=\dfrac{k(a-a_k)(a+a_k)}{k-1}$, it follows that
$$(a-a_k)(x-a_k)=\frac{k(a-a_k)^2(a+a_k)}{(k-1)(x+a_k)}\geq 0,$$

and hence we have to show that

$$x^2 + a_k^2 + xa_k + a(x + a_k) + a^2 + axa_k(x + a_k) + a^2xa_k \geq 1.$$

In order to show this inequality, we notice that

$$x^2 + a_k^2 = \frac{ka^2 + (k-2)a_k^2}{k-1} \geq \frac{ka^2}{k-1}$$

and

$$x + a_k \geq \sqrt{x^2 + a_k^2} \geq a\sqrt{\frac{k}{k-1}}.$$

Consequently, we have

$$x^2 + a_k^2 + xa_k + a(x + a_k) + a^2 + axa_k(x + a_k) + a^2xa_k \geq (x^2 + a_k^2) + a(x + a_k) + a^2 \geq$$

$$\geq \left(\frac{k}{k-1} + \sqrt{\frac{k}{k-1}} + 1\right) a^2 > 3a^2 = 1,$$

and the proof is complete. Equality holds when $a_1 = a_2 = \ldots = a_n$.

Remarks.

1. From the final solution, we can easily see that the inequality is valid for the larger condition

$$a \geq \frac{1}{\sqrt{1 + \sqrt{\frac{n}{n-1}} + \frac{n}{n-1}}}.$$

This is the largest range for a, because for $a_1 = a_2 = \cdots = a_{n-1} = x$ and $a_n = 0$ (therefore $a = x\sqrt{\frac{n-1}{n}}$), from the given inequality we get $a \geq \frac{1}{\sqrt{1 + \sqrt{\frac{n}{n-1}} + \frac{n}{n-1}}}$.

2. The special case $n = 3$ and $a = \frac{\sqrt{3}}{3}$ is a problem from Crux 2003.

120. [Vasile Cîrtoaje, Mircea Lascu] Let a, b, c, x, y, z be positive real numbers such that

$$(a + b + c)(x + y + z) = (a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = 4.$$

Prove that

$$abcxyz < \frac{1}{36}.$$

Solution:

Using the **AM-GM Inequality**, we have

$$4(ab+bc+ca)(xy+yz+zx) = \left[(a + b + c)^2 - (a^2 + b^2 + c^2)\right]\left[(x + y + z)^2 - (x^2 + y^2 + z^2)\right] =$$

$$= 20 - (a + b + c)^2(x^2 + y^2 + z^2) - (a^2 + b^2 + c^2)(x + y + z)^2 \leq$$

$$\leq 20 - 2\sqrt{(a + b + c)^2(x^2 + y^2 + z^2)(a^2 + b^2 + c^2)(x + y + z)^2} = 4,$$

therefore

$$(ab+bc+ca)(xy+yz+za) \le 1. \qquad (1)$$

By multiplying the well-known inequalities

$$(ab+bc+ca)^2 \ge 3abc(a+b+c), \quad (xy+yz+zx)^2 \ge 3xyz(x+y+z),$$

it follows that

$$(ab+bc+ca)^2(xy+yz+zx)^2 \ge 9abcxyz(a+b+c)(x+y+z),$$

or

$$(ab+bc+ca)(xy+yz+zx) \ge 36abcxyz. \quad (2)$$

From (1) and (2), we conclude that

$$1 \le (ab+bc+ca)(xy+yz+zx) \ge 36abcxyz,$$

therefore $1 \le 36abcxyz$.

To have $1 = 36abcxyz$, the equalities $(ab+bc+ca)^2 = 3abc(a+b+c)$ and $(xy+yz+zx)^2 = 3xyz(x+y+z)$ are necessary. But these conditions imply $a=b=c$ and $x=y=z$, which contradict the relations $(a+b+c)(x+y+z) = (a^2+b^2+c^2)(x^2+y^2+z^2) = 4$. Thus, it follows that $1 > 36abcxyz$.

121. [Gabriel Dospinescu] For a given $n > 2$, find the minimal value of the constant k_n, such that if $x_1, x_2, \ldots, x_n > 0$ have product 1, then

$$\frac{1}{\sqrt{1+k_n x_1}} + \frac{1}{\sqrt{1+k_n x_2}} + \cdots + \frac{1}{\sqrt{1+k_n x_n}} \le n-1.$$

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Solution:

We will prove that $k_n = \dfrac{2n-1}{(n-1)^2}$. Taking $x_1 = x_2 = \cdots = x_n = 1$, we find that $k_n \ge \dfrac{2n-1}{(n-1)^2}$. So, it remains to show that

$$\sum_{k=1}^{n} \frac{1}{\sqrt{1+\dfrac{2n-1}{(n-1)^2}x_k}} \le n-1$$

if $x_1 \cdot x_2 \cdot \cdots \cdot x_n = 1$. Suppose this inequality doesn't hold for a certain system of n numbers with product 1, $x_1, x_2, \ldots, x_n > 0$. Thus, we can find a number $M > n-1$ and some numbers $a_i > 0$ which add up to 1, such that

$$\frac{1}{\sqrt{1+\dfrac{2n-1}{(n-1)^2}x_k}} = Ma_k.$$

Thus, $a_k < \dfrac{1}{n-1}$ and we have

$$1 = \prod_{k=1}^{n} \left[\frac{(n-1)^2}{2n-1} \left(\frac{1}{M^2 a_k^2} - 1 \right) \right] \Rightarrow \left(\frac{2n-1}{(n-1)^2} \right)^n < \prod_{k=1}^{n} \left(\frac{1}{(n-1)^2 a_k^2} - 1 \right).$$

Now, denote $1-(n-1)a_k = b_k > 0$ and observe that $\sum_{k=1}^{n} b_k = 1$. Also, the above inequality becomes

$$(n-1)^{2n} \prod_{k=1}^{n} b_k \cdot \prod_{k=1}^{n} (2-b_k) > (2n-1)^n \left(\prod_{k=1}^{n} (1-b_k) \right)^2 .$$

Because from the **AM-GM Inequality** we have

$$\prod_{k=1}^{n} (2-b_k) \le \left(\frac{2n-1}{n} \right)^n ,$$

our assumption leads to

$$\prod_{k=1}^{n} (1-b_k)^2 < \frac{(n-1)^{2n}}{n^n} b_1 b_2 \dots b_n.$$

So, it is enough to prove that for any positive numbers $a_1, a_2, \dots, a_n$ the inequality

$$\prod_{k=1}^{n} (a_1 + a_2 + \dots + a_{k-1} + a_{k+1} + \dots + a_n)^2 \ge \frac{(n-1)^{2n}}{n^n} a_1 a_2 \dots a_n (a_1 + a_2 + \dots + a_n)^n$$

holds.

This strong inequality will be proved by induction. For $n = 3$, it follows from the fact that

$$\frac{\left(\prod (a+b)\right)^2}{abc} \ge \frac{\left(\frac{8}{9} \cdot \left(\sum a\right) \cdot \left(\sum ab\right) \right)^2}{abc} \ge \frac{64}{27} \left(\sum a\right)^3 .$$

Suppose the inequality is true for all systems of n numbers. Let $a_1, a_2, \dots, a_{n+1}$ be positive real numbers. Because the inequality is symmetric and homogeneous, we may assume that $a_1 \le a_2 \le \dots \le a_{n+1}$ and also that $a_1 + a_2 + \dots + a_n = 1$. Applying the inductive hypothesis we get the inequality

$$\prod_{i=1}^{n} (1-a_i)^2 \ge \frac{(n-1)^{2n}}{n^n} a_1 a_2 \dots a_n.$$

To prove the inductive step, we must prove that

$$\prod_{i=1}^{n} (a_{n+1} + 1 - a_i)^2 \ge \frac{n^{2n+2}}{(n+1)^{n+1}} a_1 a_2 \dots a_n a_{n+1} (1 + a_{n+1})^{n+1}.$$

Thus, it is enough to prove the stronger inequality

$$\prod_{i=1}^{n} \left(1 + \frac{a_{n+1}}{1-a_i} \right)^2 \ge \frac{n^{3n+2}}{(n-1)^{2n} \cdot (n+1)^{n+1}} a_{n+1} (1 + a_{n+1})^{n+1}.$$

Now, using **Huygens Inequality** and the **AM-GM Inequality**, we find that

$$\prod_{i=1}^{n}\left(1+\frac{a_{n+1}}{1-a_i}\right)^2 \geq \left(1+\frac{a_{n+1}}{\sqrt[n]{\prod_{i=1}^{n}(1-a_i)}}\right)^{2n} \geq \left(1+\frac{na_{n+1}}{n-1}\right)^{2n}$$

and so we are left with the inequality

$$\left(1+\frac{na_{n+1}}{n-1}\right)^{2n} \geq \frac{n^{3n+2}}{(n-1)^{2n}\cdot(n+1)^{n+1}}a_{n+1}(1+a_{n+1})^{n+1}$$

if $a_{n+1} \geq \max\{a_1, a_2, \ldots, a_n\} \geq \frac{1}{n}$. So, we can put $\frac{n(1+a_{n+1})}{n+1} = 1+x$, where x is nonnegative. So, the inequality becomes

$$\left(1+\frac{x}{n(x+1)}\right)^{2n} \geq \frac{1+(n+1)x}{(x+1)^{n-1}}.$$

Using **Bernoulli Inequality**, we find immediately that

$$\left(1+\frac{x}{n(x+1)}\right)^{2n} \geq \frac{3x+1}{x+1}.$$

Also, $(1+x)^{n-1} \geq 1+(n-1)x$ and so it is enough to prove that

$$\frac{3x+1}{x+1} \geq \frac{1+(n+1)x}{1+(n-1)x}$$

which is trivial.

So, we have reached a contradiction assuming the inequality doesn't hold for a certain system of n numbers with product 1, which shows that in fact the inequality is true for $k_n = \frac{2n-1}{(n-1)^2}$ and that this is the value asked by the problem.

Remark.

For $n = 3$, we find an inequality stronger than a problem given in China Mathematical Olympiad in 2003. Also, the case $n = 3$ represents a problem proposed by Vasile Cârtoaje in Gazeta Matematică, Seria A.

122. [Vasile Cîrtoaje, Gabriel Dospinescu] For a given $n > 2$, find the maximal value of the constant k_n such that for any $x_1, x_2, \ldots, x_n > 0$ for which $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$ we have the inequality

$$(1-x_1)(1-x_2)\ldots(1-x_n) \geq k_n x_1 x_2 \ldots x_n.$$

Solution:

We will prove that this constant is $(\sqrt{n}-1)^n$. Indeed, let $a_i = x_i^2$. We must find the minimal value of the expression

$$\frac{\prod_{i=1}^{n}(1-\sqrt{a_i})}{\prod_{i=1}^{n}\sqrt{a_i}}$$

when $a_1+a_2+\cdots+a_n = 1$. Let us observe that proving that this minimum is $(\sqrt{n}-1)^n$ reduces to proving that

$$\prod_{i=1}^{n}(1-a_i) \geq (\sqrt{n}-1)^n \cdot \prod_{i=1}^{n}\sqrt{a_i} \cdot \prod_{i=1}^{n}(1+\sqrt{a_i}).$$

But from the result proved in the solution of the problem 121, we find that

$$\prod_{i=1}^{n}(1-a_i)^2 \geq \left(\frac{(n-1)^2}{n}\right)^n \prod_{i=1}^{n} a_i.$$

So, it is enough to prove that

$$\prod_{i=1}^{n}(1+\sqrt{a_i}) \leq \left(1+\frac{1}{\sqrt{n}}\right)^n.$$

But this is an easy task, because from the **AM-GM Inequality** we get

$$\prod_{i=1}^{n}(1+\sqrt{a_i}) \leq \left(1+\frac{\sum_{i=1}^{n}\sqrt{a_i}}{n}\right)^n \leq \left(1+\frac{1}{\sqrt{n}}\right)^n,$$

the last one being a simple consequence of the **Cauchy-Schwarz Inequality**.

Remark.

The case $n = 4$ was proposed by Vasile Cârtoaje in Gazeta Matematică Annual Contest, 2001.

Glossary

(1) **Abel's Summation Formula**
If $a_1, a_2, ..., a_n,\ b_1, b_2, ..., b_n$ are real or complex numbers, and

$$S_i = a_1 + a_2 + ... + a_i, i = 1, 2, ..., n,$$

then

$$\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n-1} S_i(b_i - b_{i+1}) + S_n b_n.$$

(2) **AM-GM (Arithmetic Mean-Geometric Mean) Inequality**
If $a_1, a_2, ..., a_n$ are nonnegative real numbers, then

$$\frac{1}{n}\sum_{i=1}^{n} a_i \geq (a_1 a_2 ... a_n)^{\frac{1}{n}},$$

with equality if and only if $a_1 = a_2 = ... = a_n$. This inequality is a special case of the **Power Mean Inequality**.

(3) **Arithmetic Mean-Harmonic Mean (AM-HM) Inequality**
If $a_1, a_2, ..., a_n$ are positive real numbers, then

$$\frac{1}{n}\sum_{i=1}^{n} a_i \geq \frac{1}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{a_i}}$$

with equality if and only if $a_1 = a_2 = ... = a_n$. This inequality is a special case of the **Power Mean Inequality**.

(4) **Bernoulli's Inequality**
For any real numbers $x > -1$ and $a > 1$ we have $(1 + x)^a \geq 1 + ax$.

(5) **Cauchy-Schwarz's Inequality**
For any real numbers $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$

$$(a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2) \geq$$
$$\geq (a_1b_1 + a_2b_2 + ... + a_nb_n)^2,$$

with equality if and only if a_i and b_i are proportional, $i = 1, 2, ..., n$.

(6) **Cauchy-Schwarz's Inequality for integrals**
If a, b are real numbers, $a < b$, and $f, g : [a, b] \to R$ are integrable functions, then

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \left(\int_a^b f^2(x)dx\right) \cdot \left(\int_a^b g^2(x)dx\right).$$

(7) **Chebyshev's Inequality**
If $a_1 \leq a_2 \leq \ldots \leq a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers, then

1)If $b_1 \leq b_2 \leq \ldots \leq b_n$ then $\displaystyle\sum_{i=1}^n a_ib_i \geq \frac{1}{n}\left(\sum_{i=1}^n a_i\right) \cdot \left(\sum_{i=1}^n b_i\right)$;

2)If $b_1 \geq b_2 \geq \ldots \geq b_n$ then $\displaystyle\sum_{i=1}^n a_ib_i \leq \frac{1}{n}\left(\sum_{i=1}^n a_i\right) \cdot \left(\sum_{i=1}^n b_i\right)$;

(8) **Chebyshev's Inequality for integrals**
If a, b are real numbers, $a < b$, and $f, g : [a, b] \to R$ are integrable functions and having the same monotonicity, then

$$(b - a)\int_a^b f(x)g(x)dx \geq \int_a^b f(x)dx \cdot \int_a^b g(x)dx$$

and if one is increasing, while the other is decreasing the reversed inequality is true.

(9) **Convex function**
A real-valued function f defined on an interval I of real numbers is convex if, for any x, y in I and any nonnegative numbers α, β with sum 1,

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y).$$

(10) **Convexity**
A function $f(x)$ is **concave up (down)** on $[a, b] \subseteq \mathbb{R}$ if $f(x)$ lies under (over) the line connecting $(a_1, f(a_1))$ and $(b_1, f(b_1))$ for all

$$a \leq a_1 < x < b_1 \leq b.$$

A function $g(x)$ is concave up (down) on the Euclidean plane if it is concave up (down) on each line in the plane, where we identify the line naturally with $\mathbb{R}$.

Concave up and down functions are also called **convex** and **concave**, respectively.

If f is concave up on an interval $[a,b]$ and $\lambda_1, \lambda_2, ..., \lambda_n$ are nonnegative numbers wit sum equal to 1, then

$$\lambda_1 f(x_1) + \lambda_2 f(x_2) + ... + \lambda_n f(x_n) \geq f(\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n)$$

for any $x_1, x_2, ..., x_n$ in the interval $[a,b]$. If the function is concave down, the inequality is reversed. This is **Jensen's Inequality**.

(11) **Cyclic Sum**

Let n be a positive integer. Given a function f of n variables, define the cyclic sum of variables $(x_1, x_2, ..., x_n)$ as

$$\sum_{cyc} f(x_1, x_3, ..., x_n) = f(x_1, x_2, ..., x_n) + f(x_2, x_3, ..., x_n, x_1)$$

$$+ ... + f(a_n, a_1, a_2, ..., a_{n-1}).$$

(12) **Hölder's Inequality**

If r, s are positive real numbers such that $\frac{1}{r} + \frac{1}{s} = 1$, then for any positive real numbers $a_1, a_2, \ldots, a_n$,
$b_1, b_2, \ldots, b_n$,

$$\frac{\sum_{i=1}^{n} a_i b_i}{n} \leq \left(\frac{\sum_{i=1}^{n} a_i^r}{n} \right)^{\frac{1}{r}} \cdot \left(\frac{\sum_{i=1}^{n} b_i^s}{n} \right)^{\frac{1}{s}}.$$

(13) **Huygens Inequality**

If $p_1, p_2, \ldots, p_n, a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are positive real numbers with $p_1 + p_2 + \ldots + p_n = 1$, then

$$\prod_{i=1}^{n} (a_i + b_i)^{p_i} \geq \prod_{i=1}^{n} a_i^{p_i} + \prod_{i=1}^{n} b_i^{p_i}.$$

(14) **Mac Laurin's Inequality**

For any positive real numbers $x_1, x_2, \ldots, x_n$,

$$S_1 \geq S_2 \geq \ldots \geq S_n,$$

where

$$S_k = \sqrt[k]{\frac{\sum\limits_{1\le i_1<i_2<\ldots<i_k\le n} x_{i_1}\cdot x_{i_2}\cdot\ldots\cdot x_{i_k}}{\binom{n}{k}}}.$$

(15) **Minkowski's Inequality**

For any real number $r \ge 1$ and any positive real numbers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$,

$$\left(\sum_{i=1}^{n}(a_i+b_i)^r\right)^{\frac{1}{r}} \le \left(\sum_{i=1}^{n}a_i^r\right)^{\frac{1}{r}} + \left(\sum_{i=1}^{n}b_i^r\right)^{\frac{1}{r}}.$$

(16) **Power Mean Inequality**

For any positive real numbers $a_1, a_2, \ldots, a_n$ with sum equal to 1, and any positive real numbers $x_1, x_2, \ldots, x_n$, we define $M_r = (a_1x_1^r + a_2x_2^r + \ldots + a_nx_n^r)^{\frac{1}{r}}$ if r is a non-zero real and $M_0 = x_1^{a_1}x_2^{a_2}\ldots x_n^{a_n}$, $M_\infty = max\{x_1, x_2, \ldots, x_n\}$, $M_{-\infty} = min\{x_1, x_2, \ldots, x_n\}$. Then for any reals $s \le t$ we have $M_{-\infty} \le M_s \le M_t \le M_\infty$.

(17) **Root Mean Square Inequality**

If $a_1, a_2, ..., a_n$ are nonnegative real numbers, then

$$\sqrt{\frac{a_1^2+a_2^2+...+a_n^2}{n}} \ge \frac{a_1+a_2+...+a_n}{n},$$

with equality if and only if $a_1 = a_2 = ... = a_n$.

(18) **Schur's Inequality**

For any positive real numbers x, y, z and any $r > 0$, $x^r(x-y)(x-z)+y^r(y-z)(y-x)+z^r(z-x)(z-y) \ge 0$. The most common case is $r = 1$, which has the following equivalent forms:

1) $x^3+y^3+z^3+3xyz \ge xy(x+y)+yz(y+z)+zx(z+x)$;

2) $xyz \ge (x+y-z)(y+z-z)(z+x-y)$;

3) $if \quad x+y+z=1 \quad then \quad xy+yz+zx \le \dfrac{1+9xyz}{4}$.

(19) **Suranyi's Inequality**

For any nonnegative real numbers $a_1, a_2, \ldots, a_n$,

$$(n-1)\sum_{k=1}^{n}a_k^n + n\prod_{k=1}^{n}a_k \ge \left(\sum_{k=1}^{n}a_k\right)\cdot\left(\sum_{k=1}^{n}a_k^{n-1}\right).$$

(20) **Turkevici's Inequality**

For any positive real numbers x, y, z, t,

$$x^4 + y^4 + z^4 + t^4 + 2xyzt \geq x^2y^2 + y^2z^2 + z^2t^2 + t^2x^2 + x^2z^2 + y^2t^2.$$

(21) **Weighted AM-GM Inequality**

For any nonnegative real numbers $a_1, a_2, ..., a_n$, if $w_1, w_2, ..., w_n$ are nonnegative real numbers (weights) with sum 1, then

$$w_1a_1 + w_2a_2 + ... + w_na_n \geq a_1^{w_1}a_2^{w_2} \ldots a_n^{w_n},$$

with equality if and only if $a_1 = a_2 = ... = a_n$.

Further reading

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